Exercices

Exercices of Chapter 1

Ex. 1.1 *A neutral conductor is contained in a slab domain $\mathcal{D}$ consisting of two infinite parallel planes separated by L. An external uniform electrostatic field perpendicular to the planes is assumed to be instantaneously applied at time $t = 0$. Estimate within a simple Drude model the characteristic time for reaching the equilibrium configuration of the free charges.*

The conductor is assumed to be made of classical mobile electrons (charge $q = -e$, mass m) and ions fixed at the sites of a lattice, which can be viewed as a rigid background with smeared uniform charge density $c_{\mathrm{B}} = e\rho_{\mathrm{B}}$ (namely the OCP model). Within the slab geometry, all relevant quantities depend on time t and on the x-component of the position along an axis perpendicular to the planes located at $x = \pm L/2$. At $t = 0$, the electrons are at rest with an average velocity $v(x, 0) = 0$ along the x-axis and an uniform density $\rho(x, 0) = \rho_{\mathrm{B}}$. Under the action of the external uniform electric field E_{ext} with x-component E_0, the electrons start to move and they acquire the average velocity $v(x,t)$ and their density becomes $\rho(x, t) = \rho_{\mathrm{B}} + n(x, t)$. Assuming E_0 is sufficiently weak, all equations will be linearized with respect to v and n. At the local level, the Drude model provides

$$\frac{\partial v}{\partial t}(x,t) = \frac{q}{m} E(x,t) - \frac{v(x,t)}{\tau} \tag{1}$$

with a microscopic relaxation time τ ((1) is the linearized version of the Euler equation for the electron fluid with the additional volume friction force $-\rho v/\tau$). The local electric field $E(x, t)$, sum of E_0 and of the internal field created by $qn(x, t)$, satisfies the Maxwell-Gauss equation $\partial E/\partial x = 4\pi qn$. The local electrical current $j(x,t)$ is the sum of the conduction current $j_{\mathrm{cond}} = q\rho_{\mathrm{B}} v$ and of the diffusion current $j_{\mathrm{diff}} = -qD\partial n/\partial x$ with the diffusion constant D. It obeys the charge conservation law, $\partial n/\partial t + \partial j/\partial x = 0$.

A straightforward combination of the three independent equations for n,v,E yields a Poisson-Boltzmann equation for the time-Laplace transform $\hat{n}(x, s) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st} n(x, t)$, *i.e.*

$$\frac{\partial^2 \hat{n}}{\partial x^2} - \kappa^2(s)\hat{n} = 0 \,, \tag{2}$$

with $\kappa^2(s) = \kappa^2(0)(1 + s\tau)^{-1} + s/D$ and $\kappa^2(0) = 4\pi\rho_{\mathrm{B}} q^2\tau/(mD)$. Taking into account the boundary conditions which fix $\hat{n}(0, s) = 0$ and $\hat{n}(\pm L/2,s)$ ($\hat{j}(\pm L/2, s) = 0$ because the electrons cannot escape from the slab), one finds that, for a macroscopic slab with $\kappa L \gg 1$, $\hat{n}(x,s)$ is exponentially localized near the walls at $x = \pm L/2$ over the length $[\kappa(s)]^{-1}$ which carry a surface charge $\hat{\sigma_\pm}(s)$. The inverse Laplace transform of $\hat{\sigma_\pm}(s)$ is readily calculated by using its poles in the complex plane at $z = 0, -1/(2\tau) \pm i\Omega$ with $\Omega = \tau^{-1}\sqrt{\kappa^2(0)D\tau - 1/4}$,

$$\sigma_\pm(t) = \pm\frac{E_0}{4\pi}\frac{\kappa^2(0)D}{\Omega\tau}\int_0^t \mathrm{d}t'\, \mathrm{e}^{-t'/(2\tau)} \sin(\Omega t') \,. \tag{3}$$

The above derivation is heuristic since it relies on both the phenomenological friction force and the application of Fick's diffusion law at a microscopic scale. A more satisfactory approach (but much more difficult) requires the description of electrons motion *via* a suitable kinetic theory which also accounts for their quantum nature as well as the ions degrees of freedom. Nevertheless this simple Drude model does predict that the equilibrium charge density $qn(x,\infty) = q\lim_{s\to 0}[s\hat{n}(x,s)]$ is exponentially localized near the walls and perfectly screens the external field in the bulk far from the walls. In fact, this charge density is identical to expression (1.39) if one identifies $\kappa^2(0) = 4\pi\rho_{\mathrm{B}} q^2\tau/(mD)$ and $\kappa^2 = 4\pi\rho_{\mathrm{B}}\beta q^2$, *i.e.* $\tau/(mD)$ and β. The corresponding Ω then reduces to $\Omega = \omega_{\mathrm{p}}\sqrt{1 - 1/(4\omega_{\mathrm{p}}^2\tau^2)}$ with the plasma frequency $\omega_{\mathrm{p}} = (4\pi\rho_{\mathrm{B}} q^2/m)^{1/2}$. For ordinary metals at room temperature, τ is of order 10^{-14} s while ω_{p} lies between 10^{15} s^{-1} and 10^{16} s^{-1}. Hence the relaxation towards equilibrium is fast within a time scale of order 10^{-14} s and it involves many oscillations at a frequency close to ω_{p} since $\omega_{\mathrm{p}}\tau \gg 1$.

Ex. 1.2 *A conductor carrying a net charge Q is contained in a spherical domain $\mathcal{D}$ with radius R, and submitted to an uniform external field $\mathbf{E}_{\mathrm{ext}}(\mathbf{r}) = \mathbf{E}_0$. Determine the corresponding surface charge density $\sigma(\mathbf{r})$.*

Within macroscopic electrostatics, in the absence of external field, Q is uniformly distributed on the surface of the sphere with a surface charge density $Q/(4\pi R^2)$. In the presence of $\mathbf{E}_{\mathrm{ext}}(\mathbf{r}) = \mathbf{E}_0$, at equilibrium the charges are still distributed on this surface, with a surface charge density $\sigma(\mathbf{r})$ such that the total electric field vanishes inside $\mathcal{D}$, or equivalently such that the total electrostatic potential $-\mathbf{E}_0 \cdot \mathbf{r} + \varphi(\mathbf{r})$ with $\varphi(\mathbf{r})$ created by $\sigma(\mathbf{r})$, is constant for $r < R$. Hence, in spherical coordinates with the origin at the center of the sphere and Oz along $\mathbf{E}_0$, for $r < R$ $\varphi^{\mathrm{in}}(r,\theta) = \mathrm{cst} + E_0 r\cos\theta$. For $r > R$, $\varphi^{\mathrm{out}}(r,\theta)$ is an harmonic function ($\Delta\varphi^{\mathrm{out}}(\mathbf{r}) = 0$) so it can be decomposed a sum over Legendre polynomials $P_l(\cos\theta)$ ($l \geq 0$) with coefficients $\varphi_0^{\mathrm{out}}(r) = \varphi_\infty + A_0/r$, $\varphi_l^{\mathrm{out}}(r) = A_l/r^{l+1}$ ($l \geq 1$). The matching conditions at $r = R$, $\varphi^{\mathrm{in}}(R,\theta) = \varphi^{\mathrm{out}}(R,\theta)$ and $\partial(\varphi^{\mathrm{in}}(r,\theta) - \varphi^{\mathrm{out}}(r,\theta))\partial r|_{r=R} = 4\pi\sigma(\theta)$, yield $A_l = 0$ for $l > 1$ and $\sigma(\theta) = Q/(4\pi R^2) + 3E_0\cos\theta/(4\pi)$.

Ex. 1.3 *Show that the normalization constant C_{sphere} involved in formula (1.28) reduces to the expression (1.29).*

In spherical coordinates the total charge carried by $q\delta\rho(r)$ reduces to $4\pi q C_{\mathrm{sphere}}$ times the one-dimensional integral $\int_0^R \mathrm{d}r\, r\sinh(\kappa r)$. An integration by parts combined with the standard identities relating hyperbolic functions and their derivatives, provides formula (1.29) for C_{sphere}.

Ex. 1.4 *Solve the non-linear PB equation (1.46). In the limit $L \to \infty$, derive formula (1.49) for the charge distribution.*

Multiplying both sides of equation (1.46) by $\mathrm{d}\varphi/\mathrm{d}x$ and integrating from 0 to x, lead to

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 = E^2(0) + 32\pi C k_{\mathrm{B}}T\sinh^2(\beta q\varphi(x)/2) \ , \tag{4}$$

where $E(0) = -\mathrm{d}\varphi/\mathrm{d}x(0)$ is the total electrostatic field at $x = 0$. This implies that $\mathrm{d}\phi/\mathrm{d}x$ cannot vanish and it keeps a constant sign, which is negative according to the boundary condition (1.47). Then, a straightforward integration of the first-order differential equation (4) by the method of separation of variables yields ($\varphi(0) = 0$) the implicit equation relating $\varphi(x)$ to x

$$\int_0^{\beta q\varphi(x)/2} \mathrm{d}u \frac{1}{[E^2(0) + 32\pi C k_{\mathrm{B}}T\sinh^2(u)]^{1/2}} = -\frac{\beta q}{2}x \ . \tag{5}$$

The potential at the left wall $\varphi(-L/2) = \varphi_{\mathrm{W}}$ ($\varphi(L/2) = -\varphi_{\mathrm{W}}$) is eliminated in favor of C and $E(0)$ by applying (4) at $x = -L/2$ and using (1.47). The constants C and $E(0)$ are then determined by applying (5) at $x = -L/2$ and using the normalization constraint (1.44).

In the limit of a macroscopic slab, namely $L \to \infty$ while the other parameters are kept fixed, C is close to ρ while $\beta q\varphi(x)$ remains bounded according to (4) and (1.47). For $x = -L/2$ the r.h.s. of (5) diverges when $L \to \infty$: this implies that its l.h.s. must also diverge, so $E(0)$ vanishes in this limit. Equation (4) at $x = -L/2$ then gives $\sinh(\beta q\varphi_{\mathrm{W}}/2) = \beta qE_0/(2\kappa)$ with $\kappa = (8\pi\beta q^2\rho)^{1/2}$. Applying (5) at a finite distance x_{W} from the left wall ($x = -L/2 + x_{\mathrm{W}}$), one obtains

$$\int_{\beta q\varphi_{\mathrm{W}}/2}^{\beta q\varphi(x_{\mathrm{W}})/2} \mathrm{d}u \frac{1}{\sinh(u)} = -\kappa x_{\mathrm{W}} \ . \tag{6}$$

The successive variable changes $v = \sinh u$, $w = 1/v$, and $w = \sinh z$, yield

$$z_{\mathrm{W}} - z(x_{\mathrm{W}}) = -\kappa x_{\mathrm{W}} \tag{7}$$

with $\sinh z_{\mathrm{W}} = 1/\sinh(\beta q\varphi_{\mathrm{W}}/2)$ and $\sinh z(x_{\mathrm{W}}) = 1/\sinh(\beta q\varphi(x_{\mathrm{W}})/2)$. Using the inversion formula, $z = \ln(y + (1+y^2)^{1/2})$ for $y = \sinh z$, Equation (7) can be recast as the expression (1.49) for $c(x) = -4\rho q \sinh(\beta q\varphi(x)/2)\cosh(\beta q\varphi(x)/2)$ with $R(E_0)$ given by (1.50).

Ex. 1.5 *Compute the charge density induced by a test point charge e_0 pinned at a point $\mathbf{r}_0$ different from the center of the spherical box, within the linearized Poisson-Boltzmann equation (1.56).*

The electrostatic potential $\varphi(\mathbf{r})$ within the linearized Poisson-Boltzmann approach is solution of

$$\Delta\varphi(\mathbf{r}) - \kappa^2\varphi(\mathbf{r}) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0) \tag{8}$$

for $r < R$. Hence $\varphi^{\mathrm{in}}(\mathbf{r})$ is the sum of the particular solution $\varphi_S(\mathbf{r}) = e_0 \mathrm{e}^{-\kappa|\mathbf{r}-\mathbf{r}_0|}/|\mathbf{r} - \mathbf{r}_0|$ and a general solution of the homogeneous equation (1.56) in the whole sphere ($r < R$), namely (in spherical coordinates with Oz along $\mathbf{r}_0$)

$$\varphi^{\mathrm{in}}(\mathbf{r}) = \varphi_S(\mathbf{r}) + \sum_{l=0}^{\infty} C_l \frac{Z_{l+1/2}(i\kappa r)}{\sqrt{r}} P_l(\cos\theta) \tag{9}$$

where $Z_{l+1/2}(i\kappa r)$ is a Bessel function of the pure imaginary argument $i\kappa r$ of order $(l + 1/2)$ which vanishes at $r = 0$, $Z_{1/2}(i\kappa r) = \sinh(\kappa r)/\sqrt{r}$, $Z_{3/2}(i\kappa r) = [\sinh(\kappa r)/(\kappa r) - \cosh(\kappa r)]/\sqrt{r}$,... Outside the sphere, $\varphi^{\mathrm{out}}(\mathbf{r})$ is an harmonic function which remains bounded, *i.e.*

$$\varphi^{\mathrm{out}}(\mathbf{r}) = \varphi_\infty + \sum_{l=0}^{\infty} \frac{A_l}{r^{l+1}} P_l(\cos\theta) \ . \tag{10}$$

The screened potential $\varphi_S(\mathbf{r})$ can be decomposed as a sum over Legendre polynomials

$$\varphi_S(\mathbf{r}) = \sum_{l=0}^{\infty} B_l^S(r,r_0) P_l(\cos\theta) \ . \tag{11}$$

with $B_l^S(r,r_0) = \int_{-1}^{1} \mathrm{d}(\cos\theta) P_l(\cos\theta)\varphi_S(r,r_0,\cos\theta)$. Since the functions $B_l^S(r,r_0)$ are given, the *a priori* unknown coefficients C_l and A_l are determined by the matching conditions of $\varphi(\mathbf{r})$ and $\partial\varphi(\mathbf{r})/\partial r$ at $r = R$. The amplitude A_0 in (10) is nothing but the total charge e_0 inside the sphere by virtue of Gauss theorem. The first two matching conditions for the components $l = 0$ and $l = 1$ of the respective Legendre polynomials decompositions yield

$$C_0 = -\left[e_0 + R^2 \frac{\partial B_0^S}{\partial r}(R, r_0)\right] \left[\kappa R \cosh(\kappa R) - \sinh(\kappa R)\right]^{-1} \tag{12}$$

and

$$C_1 = R\left[\frac{\partial B_1^S}{\partial r}(R, r_0) + 2\frac{B_1^S(R, r_0)}{R}\right] \left[\kappa \sinh(\kappa R)\right]^{-1} \ . \tag{13}$$

while similar expressions for C_l $(l \geq 2)$ can be derived.

For a macroscopic sample, $\kappa R \gg 1$ implies $\cosh(\kappa R) \sim \sinh(\kappa R) \sim \mathrm{e}^{\kappa R}/2$. For $r_0 = \xi_0 R$, when $R \to \infty$ at fixed ξ_0 $(0 < \xi_0 < 1)$, the Legendre components $B_l^S(r, r_0)$ for r large close to R can be calculated by the method of steepest descent around the maximum of $\varphi_S(\mathbf{r})$ reached at $\theta = 0$. The corresponding expansions of C_0 and C_1 read at leading orders

$$C_0 = -2e_0 \frac{\mathrm{e}^{-\kappa R}}{\kappa R} + e_0\sqrt{2}\ \frac{\mathrm{e}^{-\kappa R(2-\xi_0)}}{\kappa R \xi_0} + ... \tag{14}$$

and

$$C_1 = -e_0\sqrt{6}\ \frac{\mathrm{e}^{-\kappa R(2-\xi_0)}}{\kappa R \xi_0} + ... \tag{15}$$

A similar calculation for any $l \geq 2$ shows that all coefficients C_l decay exponentially fast as $\mathrm{e}^{-\kappa R(2-\xi_0)}$ (except for some multiplicative powers of R). Hence all terms $(l \geq 0)$ in the Legendre decomposition (9) decay exponentially fast in the bulk, at distances from the spherical boundary large compared to the screening length κ^{-1}: this agrees with the general analysis of Section 1.4.3 and the resulting limit charge density $-\kappa^2\varphi_S(\mathbf{r})/(4\pi)$ is indeed given by formula (1.89) specified to the present case. Near the boundary, *i.e.* for $r = R - x$ with x fixed, $\varphi_S(\mathbf{r})$ vanishes exponential fast and the leading contribution to $\varphi^{\mathrm{in}}(\mathbf{r})$ arises from the term $l = 0$ in (9). The resulting charge density $c_{\mathrm{W}}(x) = e_0\kappa\mathrm{e}^{-\kappa x}/(4\pi R^2)$ is localized near the spherical wall and it is purely isotropic (anisotropic corrections decay exponentially faster), consistently with the general formula (1.95) and macroscopic electrostatics prediction for the induced surface charge density $\sigma = e_0/(4\pi R^2) = \int_0^\infty \mathrm{d}x c_{\mathrm{W}}(x)$.

Ex. 1.6 *Calculate the inverse Fourier transform of expression (1.74) for the potential $\tilde{\varphi}(\mathbf{k})$.*

The inverse Fourier transform of (1.74) reads

$$\varphi(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}}\frac{4\pi e_0}{k^2 + \kappa_{\mathrm{D}}^2} = \frac{2e_0}{\pi r}\int_0^\infty \mathrm{d}k\ \sin(kr)\ \frac{k}{k^2 + \kappa_{\mathrm{D}}^2}\ . \tag{16}$$

Take into account the parity of the integrand, $\int_0^\infty \mathrm{d}k ... = (1/2)\int_{-\infty}^{\infty}\mathrm{d}k...$, while $\sin(kr) = (\mathrm{e}^{ikr} - \mathrm{e}^{-ikr})/(2i)$. The analytic functions $f_\pm(z) = z\mathrm{e}^{\pm ikz}/(z^2+\kappa_\mathrm{D}^2)$ have simple poles at $z_1 = i\kappa_\mathrm{D}$ and $z_2 = -i\kappa_\mathrm{D} = -z_1$. Since $f_+(z)$ $(f_-(z))$ decay sufficiently fast when $|z| \to \infty$ in the upper (lower) complex plane, the integral of $f_+(z)$ $(f_-(z))$ on the real axis can be calculated *via* the theorem of residues by considering a closed contour including an upper (lower) half circle with infinite radius, namely

$$\int_{-\infty}^{\infty} \mathrm{d}k\ f_+(z) = 2i\pi \frac{z_1 \mathrm{e}^{iz_1 r}}{z_1 - z_2} \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{d}k\ f_-(z) = -2i\pi \frac{z_2 \mathrm{e}^{-iz_2 r}}{z_2 - z_1}\,. \tag{17}$$

Inserting (17) into (16) yields $\varphi(\mathbf{r}) = (2e_0/\pi r)(1/4i)(2i\pi)2(\mathrm{e}^{-\kappa_\mathrm{D} r}/2) = e_0 \mathrm{e}^{-\kappa_\mathrm{D} r}/r$.

Ex. 1.7 *Show that $\kappa_\mathrm{TF} \sim \kappa_\mathrm{D}$ when either $T \to \infty$ at fixed ρ, or $\rho \to 0$ at fixed T.*

(i) The limit $\rho \to 0$ at fixed T can be attained within the Fermi-Dirac expression (1.79) by letting $\mu \to -\infty$. The denominator in (1.79) can then be replaced by $e^{\beta(\varepsilon(\mathbf{k})-\mu)}$ yielding, after a straightforward integration of the resulting Gaussian integral over $\mathbf{k}$, $\rho \sim 2\mathrm{e}^{\beta\mu}/(2\pi\lambda_\mathrm{e}^2)^{3/2}$ with the de Broglie thermal wavelength $\lambda_\mathrm{e} = (\beta\hbar^2/m_\mathrm{e})^{1/2}$. Hence ρ indeed vanishes, and $\partial\rho_\mathrm{FD}/\partial\mu \sim \beta\rho$, so expression (1.81) of κ_TF^2 indeed reduces to κ_D^2.

(ii) The limit $T \to \infty$ at fixed ρ can be reached from (1.79) by scaling the chemical potential as $\mu \sim k_\mathrm{B}T \ln[C(2\pi\lambda_\mathrm{e}^2)^{3/2}/2]$ and letting $T \to \infty$ with a constant C. Since $\beta\mu \to -\infty$, the denominator in (1.79) can be again replaced by $e^{\beta(\varepsilon(\mathbf{k})-\mu)}$ and $\rho_\mathrm{FD} \sim 2\mathrm{e}^{\beta\mu}/(2\pi\lambda_\mathrm{e}^2)^{3/2}$ indeed goes to the constant C. Thus $\rho_\mathrm{FD}(\beta,\mu)$ takes the same form as in limit (i) implying $\kappa_\mathrm{TF} \sim \kappa_\mathrm{D}$. Since the degeneracy parameter $\rho\lambda_\mathrm{e}^3$ vanishes in both limits (i) and (ii), the system becomes classical and the asymptotic forms of $\rho_\mathrm{FD}(\beta,\mu)$ are identical as it should be.

Ex. 1.8 *Determine the charge density induced by a test charge in the quantum OCP at zero temperature by using the Thomas-Fermi density functional theory.*

As explained in Section I.3, in the framework of density functional theory, the quantum OCP can be seen as an electron gas submitted to the external potential created by the rigid background with uniform charge density $e\rho_\mathrm{B}$. Within Thomas-Fermi theory applied to this electron gas, one starts from the variational equation (18.85) where G is replaced by the zero-temperature ideal Thomas-Fermi functional $E_\mathrm{TF}^\mathrm{id}$ given by (I.27). Here the external electrostatic potential $\varphi_\mathcal{Y}(\mathbf{r})$ seen by the electrons is the sum of the potential e_0/r created by the test charge e_0 pinned at the origin plus the potential $e\rho_\mathrm{B}\int \mathrm{d}\mathbf{r}'|\mathbf{r}'-\mathbf{r}|^{-1}$ created by the background. This leads to

$$(3\pi^2)^{2/3}\frac{\hbar^2}{2m_\mathrm{e}}[\rho(\mathbf{r})]^{2/3} = \mu + e\varphi(\mathbf{r}) \tag{18}$$

where $\varphi(\mathbf{r})$ is the total electrostatic potential created by e_0 plus the induced charge density $c(\mathbf{r}) = -e[\rho(\mathbf{r}) - \rho_\mathrm{B}]$. At large distances $\rho(\mathbf{r})$ goes to the uniform density $\rho = \rho_{\mathrm{B}}$ while $\varphi(\mathbf{r})$ vanishes. For an infinitesimal e_0 the deviation $(\rho(\mathbf{r}) - \rho)$ is small compared to ρ, hence the l.h.s. of (18) can be linearized yielding

$$c(\mathbf{r}) = -\frac{3^{1/3} m_{\mathrm{e}} e^2 \rho^{1/3}}{\pi^{4/3} \hbar^2} \varphi(\mathbf{r}) \, . \tag{19}$$

One recovers the closure equation (1.80) with $\kappa_{\mathrm{TF}}^2 = (m_{\mathrm{e}} e^2/\hbar^2)(192\rho/\pi)^{1/3}$ which is equal to $(4\pi e^2)\partial\rho_{\mathrm{FD}}/\partial\mu$ (equation (1.81)) calculated at zero temperature ($\beta = \infty$) with $\rho_{\mathrm{FD}}(\infty,\mu) = (2m_{\mathrm{e}}\mu/\hbar^2)^{3/2}/(3\pi^2)$. Thus, at zero temperature, Thomas-Fermi density functional theory exactly provides, as expected, the same induced charge density as that derived in Section 1.3.2.3.

Ex. 1.9 *Show that the typical size of quantum position fluctuations are small compared to the screening length for either nearly classical or strongly degenerate conditions.*

The typical size of quantum position fluctuations can be estimated as $\xi_2 = \int \mathrm{d}\mathbf{r}\; r^2 D^{(1)}(\mathbf{0},\mathbf{r}) / \int \mathrm{d}\mathbf{r}\; D^{(1)}(\mathbf{0},\mathbf{r})$ where $D^{(1)}(\mathbf{0},\mathbf{r})$ is the off-diagonal part of the equilibrium one-body density matrix. In terms of the Fourier transform $\tilde{D}^{(1)}(\mathbf{k})$, ξ_2 is proportional to the ratio of the coefficient of the k^2-term in the small k expansion of $\tilde{D}^{(1)}(\mathbf{k})$ divided by $\tilde{D}^{(1)}(\mathbf{0})$. The degeneracy effects are controlled by the parameter $\theta = \lambda_{\mathrm{e}}/a$. The nearly classical regime, with $\theta \ll 1$ can be obtained at fixed T by letting $\rho \to 0$. In this regime the screening length λ_{S} is close to the Debye length $\lambda_{\mathrm{D}} \gg a$. The strongly degenerate regime, with $\theta \gg 1$, can be attained at fixed T by letting $\rho \to \infty$: λ_{S} then is close to the Thomas-Fermi screening length $\lambda_{\mathrm{TF}} \propto (a a_{\mathrm{B}})^{1/2}$ (Ex. 1.8). It turns out that in both regimes, $\theta \ll 1$ or $\theta \gg 1$, $D^{(1)}(\mathbf{0},\mathbf{r})$ is close to its ideal counterpart because interaction effects are small, *i.e.* $e^2/a \ll k_{\mathrm{B}}T$ or $e^2/a \ll \hbar^2 k_{\mathrm{F}}^2/(2m_{\mathrm{e}})$ with the Fermi wave number $k_{\mathrm{F}} = (3\pi^2\rho)^{1/3} \propto 1/a$. Hence $\tilde{D}^{(1)}(\mathbf{k})$ can be approximated by twice (because of spin contribution) the Fermi-Dirac distribution yielding

$$\xi_2 = 3\lambda_{\mathrm{e}}^2 \frac{e^{-\beta\mu}}{[e^{-\beta\mu} + 1]} \, . \tag{20}$$

(i) $\theta \ll 1$: the limit $\rho \to 0$ at fixed T corresponds to $\mu \to -\infty$ (see Ex. 1.7). Hence expression (20) provides $\xi_2 \sim 3\lambda_{\mathrm{e}}^2$, and one indeed finds $\sqrt{\xi_2} \ll \lambda_{\mathrm{D}}$. Note that in real space $D^{(1,\mathrm{id})}(\mathbf{0},\mathbf{r})$ is proportional to the Gaussian $\mathrm{e}^{-r^2/(2\lambda_{\mathrm{e}}^2)}$.

(ii) $\theta \gg 1$: the limit $\rho \to \infty$ at fixed T corresponds to $\mu \to \infty$ yielding $\xi_2 \sim 3\lambda_{\mathrm{e}}^2 e^{-\beta\mu}$. Since μ is close to the Fermi energy $\hbar^2 k_{\mathrm{F}}^2/(2m_{\mathrm{e}})$, $\sqrt{\xi_2}/a$ vanishes exponentially fast as $\sqrt{3}(\lambda_{\mathrm{e}}/a)\mathrm{e}^{-(9\pi/4)^{2/3}(\lambda_{\mathrm{e}}/a)^2/2}$. Using $\lambda_{\mathrm{TF}}/a \propto (a_{\mathrm{B}}/a)^{1/2} \gg 1$, this implies $\sqrt{\xi_2} \ll \lambda_{\mathrm{TF}}$. There is no simple expression in real space for $D^{(1,\mathrm{id})}(\mathbf{0},\mathbf{r})$ but its large-r behavior can be determined *via* a straightforward application of the theorem residues, similarly to the calculation of Ex. 1.6, namely $D^{(1,\mathrm{id})}(\mathbf{0},\mathbf{r}) \sim -k_{\mathrm{F}}/(\pi^2 r^2)\cos(k_{\mathrm{F}} r)\mathrm{e}^{-r/(\lambda_{\mathrm{e}}^2 k_{\mathrm{F}})}$ when $r \to \infty$. The Friedel oscillations with spatial period $2\pi/k_{\mathrm{F}}$, which emerge in the zero-temperature expression $D_{T=0}^{(1,\mathrm{id})}(\mathbf{0},\mathbf{r}) = [\sin(k_{\mathrm{F}} r)/r - k_{\mathrm{F}}\cos(k_{\mathrm{F}} r)]/(\pi^2 r^2)$, are exponentially damped on the very large scale length $\lambda_{\mathrm{e}}^2 k_{\mathrm{F}} \gg a$.

Exercices of Chapter 2

Ex. 2.1 *Derive of the second moment condition (2.18).*

The first expression follows from (2.12). Expanding (2.14) to second order gives (the linear term does not contribute because of space inversion invariance of $S(\mathbf{r})$)

$$\tilde{S}(\mathbf{k}) \sim -\frac{1}{2}\int \mathrm{d}\mathbf{r}(\mathbf{k}\cdot\mathbf{r})^2 S(\mathbf{r}) = -\frac{1}{2}\sum_{i=1}^{\nu}\left(k_i^2\int \mathrm{d}\mathbf{r}\; r_i^2 S(\mathbf{r})\right)$$
$$= -\frac{|\mathbf{k}|^2}{2}\int \mathrm{d}\mathbf{r}\; r_i^2 S(\mathbf{r}) = -\frac{|\mathbf{k}|^2}{2\nu}\int \mathrm{d}\mathbf{r}|\mathbf{r}|^2 S(\mathbf{r}) \tag{21}$$

since averages of cross terms $r_i r_j$, $i \neq j$ vanish, and those of the diagonal terms r_i^2 have the same value because of spherical symmetry.

Ex. 2.2 *Extract the contributions of coincident points in $\langle\hat{\rho}(\gamma_1\mathbf{r}_1)\hat{\rho}(\gamma_2\mathbf{r}_2)\hat{\rho}(\gamma_3\mathbf{r}_3)\rangle$. Derive the general rule.*

With the abbreviated notation $\gamma_i\mathbf{r}_i = i$

$$\langle\hat{\rho}(1)\hat{\rho}(2)\hat{\rho}(3)\rangle = \rho(1,2,3) + \delta_{12}\rho(1,3) + \delta_{13}\rho(2,3) + \delta_{23}\rho(12) + \delta_{12}\delta_{23}\rho(1).$$

In general, consider all partitions of $1, 2, \ldots, n$.

Ex. 2.3 *(i) Calculate the two point correlation, the structure function and the electrical susceptibility in the mean field approximation. Show that the particle-charge structure function obeys the charge sum rule and that the Stillinger-Lovett condition is verified in this approximation. (ii) Calculate the mean field electrostatic pressure.*

(i) Consider the conditional particle distribution $\frac{\rho(\gamma\mathbf{r}|\gamma'\mathbf{r}')}{\rho_\gamma}$ around a specified charge of the system, say $e_{\gamma'}$ located at $\mathbf{r}', \mathbf{r} \neq \mathbf{r}'$. One can identify this distribution to the screening cloud surrounding an external charge e_0, setting $e_0 = e_{\gamma'}$ in the mean field results (1.76). This yields the mean field expression (2.38) of the two particle distribution for $\mathbf{r} \neq \mathbf{r}'$. Introducing this expression in the structure function (2.31) leads to

$$S(\mathbf{r}-\mathbf{r}') = \frac{\kappa_D^2}{4\pi\beta}\left[-\kappa_D^2\frac{\mathrm{e}^{-\kappa_D|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \delta(\mathbf{r}-\mathbf{r}')\right] \tag{22}$$

and finally to (2.39) after Fourier transform. The small $\mathbf{k}$ expansion gives immediately (2.17) and (2.20).

(ii) Substitute the mean-field correlation (2.38) in (2.41) and calculate the integral.

Ex. 2.4 *Show that, within linear response theory, $\rho(\mathbf{r}_2)h_{MF}(\mathbf{r}_1,\mathbf{r}_2)$ is the change in density at $\mathbf{r}_2$ when a particle is added at $\mathbf{r}_1$.*

An additional charge e at point $\mathbf{r}_1$ in the OCP causes the additional potential energy (2.4) with e_0 replaced by e and the induced charge in (2.7) is to be identified with the conditional distribution of charges $\frac{e\rho(\mathbf{r}_1|\mathbf{r}_2)}{\rho(\mathbf{r}_1)}$ around it . Then (2.66) follows from the linear response formula, and the definition of $h(\mathbf{r}_1,\mathbf{r}_2)$. Note that the results of Section 2.1.1 are exact in the limit $e_0 \to 0$ whereas (2.66) remains an approximation (equivalent to mean field) since the system's charge e is not infinitesimal.

Ex. 2.5 *Calculate the OCP pair-correlation in the vicinity of a non-polarizable wall at $x = 0$ in the mean field approximation, assuming that the density is a step function $\rho(x) = \rho, x \geq 0, \rho(x) = 0, x < 0$. Show that the screening cloud satisfies the perfect screening sum rule and has a dipole moment equal to $-(e/\kappa_D)\mathrm{e}^{-\kappa_D x}$.*

HINT: First transform the integral equation (2.66) into Poisson-Boltzmann differential equations in the regions $x_1, x_2 > 0$ and $x_1 > 0$, $x_2 < 0$. Solve them in the two-dimensional Fourier representation $\tilde{h}(x_1, x_2, \mathbf{k})$ with the condition that $\tilde{h}(x_1, x_2, \mathbf{k})$ and $\frac{\partial}{\partial x_2}\tilde{h}(x_1, x_2,\mathbf{k})$ are continuous on the plane $x_2 = 0$ ($\mathbf{k}$ a two-dimensional wave vector on the plane of the wall), see calculations in Section 3 of [4].

Ex. 2.6 *Assuming fast-decay properties of the bulk structure function $S(\mathbf{r})$, establish formulae (2.93) and (2.113).*

Using space inversion invariance $x - x' \to x' - x$, $S^b(x, \mathbf{y}) = S^b(-x, \mathbf{y})$, the l.h.s of (2.93) can as well be written, after integration by part, as

$$\begin{aligned} \int_0^\infty \mathrm{d}x' \int_{x'}^\infty \mathrm{d}x x \int \mathrm{d}\mathbf{y} S^b(x, \mathbf{y}) &= \int_0^\infty \mathrm{d}x \int \mathrm{d}\mathbf{y} x^2 S^b(x, \mathbf{y}) \\ &= 1/2 \int_{-\infty}^\infty \mathrm{d}x \int \mathrm{d}\mathbf{y} x^2 S^b(x, \mathbf{y}). \end{aligned} \tag{23}$$

The proof of (2.113) is similar.

Ex. 2.7 *Derive the formula (2.121) and find the depolarization tensor of a spherical domain.*

It is convenient to define a susceptibility χ_E as the response of the polarization density to the total field E : $\mathcal{P} = \chi_E E$. Since $D = \epsilon E = E + 4\pi\mathcal{P}$, one has $\epsilon = 1 + 4\pi\chi_E$. From (2.119) and (2.120), $E = (1 - 4\pi\chi_\mathcal{D} T_\mathcal{D})E_0$. Thus $\mathcal{P} = \chi_\mathcal{D} E_0 = \chi_E(1 - 4\pi\chi_\mathcal{D} T_\mathcal{D})E_0$ implying $\chi_E = \frac{\chi_\mathcal{D}}{1-4\pi\chi_\mathcal{D} T_\mathcal{D}}$ and hence the formula (2.121) for ϵ. Because of isotropy, $T^{ij}_{\text{sphere}} = \delta_{ij} T_{\text{sphere}}$. Thus $\sum_{i=1}^3 T^{ii}_{\text{sphere}} = -\frac{1}{4\pi}\int_{\text{sphere}} \mathrm{d}\mathbf{r}\Delta(\frac{1}{r}) = \int_{\text{sphere}} \mathrm{d}\mathbf{r}\delta(\mathbf{r}) = 1$ leading to $T^{ij}_{\text{sphere}} = \frac{\delta_{ij}}{3}$.

Ex. 2.8 *Show that Poisson's equation (2.168) with a polarization charge density $c(\mathbf{r}, \omega) = c(\mathbf{y}, \omega)\delta(x)$ concentrated on the plane $x = 0$ ($\mathbf{r} = x, \mathbf{y}$) admits solutions with an electric field localized near the interface.*

It is easily verified that

$$\Phi_{\mathbf{q}}(x, \mathbf{y}) = \frac{2\pi}{q} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{y}} \mathrm{e}^{-q|x|} \tag{24}$$

is an elementary solution of $\boldsymbol{\nabla}^2 \Phi_{\mathbf{q}}(x, \mathbf{y}) = -4\pi\delta(x)$ ($q = |\mathbf{q}|$, $\mathbf{q}$ a two dimensional wave vector in the interface). Hence the solution of (2.168) is the superposition

$$\Phi(x, \mathbf{y}, \omega) = \int \mathrm{d}^2\mathbf{q} c(\mathbf{q}, \omega) \Phi_{\mathbf{q}}(x, \mathbf{y}) \tag{25}$$

with $c(\mathbf{q}, \omega)$ the two dimensional Fourier transform of the surface charge density. Since $\Phi_{\mathbf{q}}(x, \mathbf{y})$ decays aways from the surface, so do the potential and the electric field.

Ex. 2.9 *Derive the non-retarded surface plasmon condition $\epsilon_1(\omega) + \epsilon_2(\omega) = 0$. Adopting the bulk plasmon resonance formula for the response of metallic media (the Drude formula), recover the surface plasmon oscillations (2.154), (2.159) and (2.162).*

In the model (2.169),

$$\begin{aligned} \mathbf{D}(x, \mathbf{y}, \omega) &= \epsilon_1(\omega)\mathbf{E}(x, \mathbf{y}, \omega), \quad x > 0 \\ &= \epsilon_2(\omega)\mathbf{E}(x, \mathbf{y}, \omega), \quad x < 0 \end{aligned}$$

Since $\mathbf{D}(x, \mathbf{y}, \omega)$ is divergence free, its normal component is continuous across the interface

$$\epsilon_1(\omega) E_x(x = 0^+, \mathbf{y}, z = 0^+, \omega) = \epsilon_2(\omega) E_x(x = 0^-, \mathbf{y}, \omega)$$

On the other hand it follows from (24) and (25) that $E_x(x, \mathbf{y}, \omega) = -\frac{\partial}{\partial z}\Phi(x, \mathbf{y}, \omega)$ changes its sign, $E_x(x = 0^+, \mathbf{y}, \omega) = -E_x(x = 0^-, \mathbf{y}, \omega)$. This gives the surface plasmon condition

$$\epsilon_1(\omega) + \epsilon_2(\omega) = 0 \tag{26}$$

Using the relation $\epsilon(\omega) = \frac{1}{1+\chi(\omega)}$ between the dielectric function and the susceptibility, the resonance frequencies solutions of (26) are given by (2.159) for a metal-insulator interface $\left(\chi_1(\omega) = \frac{\omega_p^2}{\omega^2 - \omega_p^2}, \epsilon_2 = \epsilon_w\right)$ and by (2.162) for a metal-metal interface $\left(\chi_1(\omega) = \frac{(\omega_p^+)^2}{\omega^2 - (\omega_p^+)^2}, \ \chi_2(\omega) = \frac{(\omega_p^-)^2}{\omega^2 - (\omega_p^-)^2}\right)$.

Exercices of Chapter 3

Ex. 3.1 *Show that the region (b) in (3.16) does not contribute to the charge fluctuation.*

Using spherical coordinates for $r^2 \dots r^\nu$ with $u = \sqrt{(r^2)^2 + \cdots + (r^\nu)^2}$ and setting $r^1 = -Lx$ (c_ν results of the angular integration)

$$\begin{aligned}\int_{-2L}^{-L} \mathrm{d}r^1 \int \mathrm{d}r^2 \cdots \mathrm{d}r^\nu |G(\mathbf{r})| &= c_\nu L \int_1^2 \mathrm{d}x \int_0^\infty \mathrm{d}u\; u^{\nu-2} |G(\sqrt{(Lx)^2 + u^2})| \\ &= c_\nu L \int_1^2 \mathrm{d}x \int_0^L \mathrm{d}u\; u^{\nu-2} |G(\sqrt{(Lx)^2 + u^2})| \\ &\quad + c_\nu L \int_1^2 \mathrm{d}x \int_L^\infty \mathrm{d}u\; u^{\nu-2} |G(\sqrt{(Lx)^2 + u^2})| \end{aligned} \tag{27}$$

Assuming exponential decay of $G(r) \sim \exp(-\kappa\sqrt{(Lx)^2 + u^2})$, one can set $|G(\sqrt{(Lx)^2 + u^2})| \leq c \exp(-\kappa Lx)$ in the first integral of the second line of (27) and $|G(\sqrt{(Lx)^2 + u^2})| \leq c' \exp(-\kappa u)$ in the second one. We then see that both terms decay as $\exp(-\kappa L)$ times powers of L.

Ex. 3.2 *Use a linear response argument to find the behavior of the potential fluctuations $\langle \hat{\psi}(\mathbf{r}) | \hat{\psi}(\mathbf{r}') \rangle_{\mathcal{D}}$ in a finite volume system when $|\mathbf{r} - \mathbf{r}'|$ is large compared to the screening length, but small with respect to the system size.*

In presence of an external charge e_0 at $\mathbf{r}'$, the average potential in the perturbed fluid is related to the unperturbed potential fluctuations by (see (A.4))

$$\left\langle \hat{\psi}(\mathbf{r}) \right\rangle_{\Lambda, e_0} - \left\langle \hat{\psi}(\mathbf{r}) \right\rangle_{\Lambda} = -\beta \left\langle \hat{\psi}(\mathbf{r}) | \hat{\psi}(\mathbf{r}') \right\rangle_{\Lambda}$$

Here $\hat{\psi}(\mathbf{r})$ is the potential due to the particles in the fluid, not including that of e_0. This external charge is surrounded by a screening cloud of charge $-e_0$, whereas, for an insulated conductor, a charge e_0 spreads on the walls. This surface charge creates a constant potential e_0/C, where C is the capacitance. Thus, for $\mathbf{r}$ larger than the screening length, but not close to the boundary, $\langle \hat{\psi}(\mathbf{r}) \rangle_{\Lambda, e_0} - \langle \hat{\psi}(\mathbf{r}) \rangle_{\Lambda} \sim -e_0/|\mathbf{r} - \mathbf{r}'| + e_0/C$. For a grounded conductor no surface charge appears. If the system become infinitely large $1/C$ goes to 0 and on retrieves (3.28).

Ex. 3.3 *Show that the electrostatic pressure calculated from the electric field fluctuations (3.34)* is the same as that found from the virial formula (2.40).

As a consequence of isotropy, the averages of the products of coordinates $x_i x_j, y_i y_j, z_i z_j$ involved in the formula (3.34) have the same value, so that, using also translation invariance (setting $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$)

$$P_{\mathrm{elec}} = \frac{1}{24\pi} \int \mathrm{d}\mathbf{r} \left[\int \mathrm{d}\mathbf{r}_1 \frac{\mathbf{r}_1 \cdot (\mathbf{r}_1 - \mathbf{r})}{|\mathbf{r}_1|^3 |\mathbf{r}_1 - \mathbf{r}|^3} \right] \sum_{\gamma_1, \gamma_2 = 1}^{\mathcal{S}} e_{\gamma_1} e_{\gamma_2} \rho(\gamma_1, \mathbf{r} | \gamma_2, 0) \tag{28}$$

The $\mathbf{r}_1$ integral is equal to

$$\int \mathrm{d}\mathbf{r}_1 \nabla_1 \left(\frac{1}{|\mathbf{r}_1|} \right) \cdot \nabla_1 \left(\frac{1}{|\mathbf{r}_1 - \mathbf{r}|} \right) = -\int \mathrm{d}\mathbf{r}_1 \nabla_1^2 \left(\frac{1}{|\mathbf{r}_1|} \right) \frac{1}{|\mathbf{r}_1 - \mathbf{r}|} = \frac{4\pi}{|\mathbf{r}|} \tag{29}$$

which yields (2.41).

Ex. 3.4 *Establish the properties (i), (ii) and (iii) of* $E_x(0,\mathbf{r})$.

Write explicitly the arguments of the potential in (3.45):

$$
\begin{aligned}
v_{\rm slab}(\mathbf{r},\mathbf{r}') &= v_c(\sqrt{|\mathbf{y}-\mathbf{y}'|^2+(x-x')^2})) - v_c(\sqrt{|\mathbf{y}-\mathbf{y}'|^2+(x+x')^2}))\\
&+\sum_{n=1}^{\infty} v_c(\sqrt{|\mathbf{y}-\mathbf{y}'|^2+(x+2nL-x')^2}) - \sum_{n=1}^{\infty} v_c(\sqrt{|\mathbf{y}-\mathbf{y}'|^2+(x-2nL+x')^2})\\
&+\sum_{n=1}^{\infty}(v_c(\sqrt{|\mathbf{y}-\mathbf{y}'|^2+(x-2nL-x')^2}) - \sum_{n=1}^{\infty}(v_c(\sqrt{|\mathbf{y}-\mathbf{y}'|^2+(x+2nL+x')^2})\\
&= A-B+I-II+III-IV
\end{aligned} \tag{30}
$$

(i) Boundary conditions : For $x=L$, the arguments in the terms I to IV become respectively $\sqrt{|\mathbf{y}-\mathbf{y}'|^2+((2n+1)L-x')^2}$, $\sqrt{|\mathbf{y}-\mathbf{y}'|^2+((2n-1)L+x')^2}$, $\sqrt{|\mathbf{y}-\mathbf{y}'|^2+((2n-1)L-x')^2}$, $\sqrt{|\mathbf{y}-\mathbf{y}'|^2+((2n+1)L+y')^2}$. With the identity $\sum_{n=1}^{\infty} f(2n-1) = f(1)+\sum_{n=1}^{\infty} f(2n+1)$ one finds $III-I = v_c(\mathbf{r}-\mathbf{r}')|_{x=L}$, $IV-II = v_c(\mathbf{r}^*-\mathbf{r}')|_{x=L}$. This compensates the terms A and B. Finally the term A is the only one that contributes to $\Delta_{\mathbf{r}} v_{\rm slab}(\mathbf{r}-\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$ for $0\le x,x'\le L$, all the other terms are not singular in $0\le x,x'\le L$ and their Laplacian vanishes. Under the exchange of $\mathbf{r}$ and $\mathbf{r}'$, the terms A,B,II,IV are invariant, and I transforms into III.

(ii) A computation yields

$$
\begin{aligned}
E_x(0,\mathbf{r}) &= -\frac{\partial}{\partial x'} v_{\rm slab}(\mathbf{r}',\mathbf{r})\Big|_{\mathbf{r}'=0} = -2\frac{x}{(|\mathbf{y}|^2+x^2)^{\nu/2}}\\
&-2\sum_{n=1}^{\infty}\left[\frac{2nL+x}{(|\mathbf{y}|^2+(2nL+x)^2)^{\nu/2}} - \frac{2nL-x}{(|\mathbf{y}|^2+(2nL-x)^2)^{\nu/2}}\right]
\end{aligned} \tag{31}
$$

Then use the expansion

$$
\frac{a+x}{(b^2+(a+x)^2)^{\nu/2}} = \frac{1}{a^{\nu-1}} + \frac{(1-\nu)\;x}{a^{\nu}} + \frac{\left((\nu^2-\nu)\;x^2-b^2\,\nu\right)}{2a^{\nu+1}} + \cdots, \quad a\to\infty \tag{32}
$$

(iii) Use

$$
a\int_0^{\infty} \mathrm{d}y \frac{2\pi y}{(a^2+y^2)^{\frac{3}{2}}} = \frac{2\pi a}{|a|} = 2\pi, \quad a\int_{-\infty}^{\infty} \mathrm{d}y \frac{1}{a^2+y^2} = \frac{\pi}{2}\frac{a}{|a|} = \frac{\pi}{2}, \quad a>0 \tag{33}
$$

and $2nL\pm y>0, n\ge 1, 0\le y\le L$.

Ex. 3.5 *Calculate the electrostatic contribution to the pressure for a Coulomb system in a half space limited by a metallic wall.*

This contribution is given by the electrostatic pressure term in (3.51). According to (3.46) and using translation invariance in the $\mathbf{y}$ direction

$$< (\hat{E}_x(0))^2 >=$$
$$4\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}x' \int \mathrm{d}^2\mathbf{y} \left[\int \mathrm{d}^2\mathbf{y}' \frac{x}{(|\mathbf{y}+\mathbf{y}'|^2+x^2)^{3/2}} \frac{x'}{(|\mathbf{y}'|^2+x'^2)^{3/2}}\right] \langle \hat{c}(\mathbf{y},x)\hat{c}(0,x')\rangle \tag{34}$$

The square bracket is calculated with the help of the convolution of Fourier transforms. With $\frac{x}{(|\mathbf{y}|^2+x^2)^{3/2}} = \frac{1}{2\pi}\int \mathrm{d}^2\mathbf{k}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{y}}\mathrm{e}^{-|\mathbf{k}|x}$ one finds the form of the electrostatic pressure

$$\begin{aligned}
-\frac{1}{8\pi}\left\langle(\hat{E}_x(0))^2\right\rangle &= -\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}x' \int \mathrm{d}^2\mathbf{y} \frac{x+x'}{(|\mathbf{y}|^2+(x+x'))^{3/2}} \langle \hat{c}(\mathbf{y},x)\hat{c}(0,x')\rangle \\
&= -\sum_\gamma e_\gamma^2 \int_0^\infty \mathrm{d}x \frac{\rho(\gamma,x)}{(2x)^2} \\
-\sum_\gamma\sum_{\gamma'} e_\gamma e_{\gamma'} &\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}x' \int \mathrm{d}^2\mathbf{y} \frac{x+x'}{(|\mathbf{y}|^2+(x+x'))^{3/2}} \rho(\gamma,\mathbf{y},x,\gamma',0,x')
\end{aligned} \tag{35}$$

To avoid divergences at $x = x' = 0$, one must take into account the short range repulsion, e.g. replace point charges by hard spheres. The first term in the r.h.s. of (35) is the pressure due to a particle with its image at the wall, and the second term represents the pressure due to the interaction of charges with different images.

If the plate at $x = 0$ carries a surface charge σ, one must add in (3.42) the corresponding external field $E_{\mathrm{ext}} = 4\pi\sigma$. Equation (3.48) becomes $\langle \hat{E}_x(0)\rangle = 4\pi\sigma$ and this gives rise to the additional term $2\pi\sigma^2$ in (35).

Exercices of Chapter 5

Ex. 5.1 *Calculate the self energy of a ball of radius R.*

HINT : One recalls the potential of a uniformly charged sphere of radius R and volume $|B_R| = \frac{4\pi R^3}{3}$ located at the origin with total charge q

$$\phi_R(\mathbf{x}) = \frac{q}{|\mathbf{x}|}, \quad |\mathbf{x}| \geq R, \qquad \phi_R(\mathbf{x}) = \frac{q}{2R}\left(3 - \frac{|\mathbf{x}|^2}{R^2}\right), \quad |\mathbf{x}| \leq R \tag{36}$$

Calculate the integral

$$u^{\mathrm{self}} = \frac{q}{|B_R|}\int_{|\mathbf{x}|\leq R} \mathrm{d}\mathbf{x} \frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right) = \frac{6q^2}{5R} \tag{37}$$

Ex. 5.2 *Calculate the potential energy of a ball of radius R in the uniform charge density background.*

According to (36), the interaction of a sphere of radius R at origin with the background charge density c_B spread in Λ is

$$c_B \int_\Lambda \mathrm{d}\mathbf{x}\phi_R(\mathbf{x}) = c_B q \int_{\Lambda, |\mathbf{x}| \geq R} \mathrm{d}\mathbf{x} \frac{1}{|\mathbf{x}|} + c_B q \int_{V, |\mathbf{x}| \leq R} \mathrm{d}\mathbf{x} \frac{1}{2R}\left(3 - \frac{|\mathbf{x}|^2}{R^2}\right) \quad (38)$$

If the sphere is inside Λ, the second integral is equal to $2\pi R^2 - 2\pi \frac{R^2}{5}$. Thus the difference (5.12) for $\mathbf{x}_i = 0$ equals

$$c_B q \int_{|\mathbf{x}| \leq R} \mathrm{d}\mathbf{x} \frac{1}{|\mathbf{x}|} - c_B q \left(2\pi R^2 - 2\pi \frac{R^2}{5}\right) = -|c_B q| 2\pi \frac{R^2}{5} \quad (39)$$

If the overlap of the ball with the background is not complete, (39) is a lower bound.

Ex. 5.3 *Potential energy of charged balls.*

HINT : For i) set the z-axis along the centers of the spheres and use bipolar coordinates

$$r_1 = \sqrt{\rho^2 + (z + r/2)^2}, \qquad r_1 = \sqrt{\rho^2 + (z - r/2)^2}, \qquad \rho = \sqrt{x^2 + y^2}$$

and φ the angle around the z-axis, $\qquad \mathrm{d}\varphi \rho \mathrm{d}\rho \mathrm{d}z = \frac{r_1 r_2}{r} \mathrm{d}\varphi \mathrm{d}r_1 \mathrm{d}r_2 \quad (40)$

According to (36), the energy of the ball B_2 of charge density $c_R = \frac{3q}{4\pi R^3}$ in the potential $\phi_R(\mathbf{x})$ due to B_1 can be written in bipolar coordinates (the system is invariant under rotations around the z-axis) as :

$$\begin{aligned} U_{12}(r) &= c_R \int_{B_2} \mathrm{d}\mathbf{x}\phi_R(\mathbf{x}) \\ &= \frac{3q^2}{2rR^3}\left[\int_{B_1 \wedge B_2} \mathrm{d}r_1 \mathrm{d}r_2 r_1 r_2 \frac{1}{2R}\left(3 - \frac{r_1^2}{R^2}\right) + \int_{B_2/B_1} \mathrm{d}r_1 \mathrm{d}r_2 \, r_2\right] \end{aligned} \quad (41)$$

The ranges of the variables r_1, r_2 are

in $B_1 \wedge B_2$: $\quad r - R \leq r_1 \leq R, \quad r - r_1 \leq r_2 \leq R$
in B_2/B_1 : $\quad R \leq r_1 \leq r, \quad r - r_1 \leq r_2 \leq R$ and $r \leq r_1 \leq r + R, r_1 - r \leq r_2 \leq R$ (42)

leading to the integrals

$$\begin{aligned} U_{12}(r) &= \frac{3q^2}{2rR^3}\left[f_{B_1 \wedge B_2}(r) + f_{B_2/B_1}(r)\right] \quad (43) \\ f_{B_1 \wedge B_2}(r) &= \int_{r-R}^{R} \mathrm{d}r_1 r_1 \frac{1}{2R}\left(3 - \frac{r_1^2}{R^2}\right) \int_{r - r_1} R \mathrm{d}r_2 \, r_2 \\ f_{B_2/B_1}(r) &= \int_R^r \mathrm{d}r_1 \int_{r-r_1}^{R} \mathrm{d}r_2 \, r_2 + \int_r^{r+R} \mathrm{d}r_1 \int_{r_1 - r}^{R} \mathrm{d}r_2 \, r_2 \end{aligned}$$

The integrals are elementary. Regrouping terms gives

$$\begin{aligned} f_{B_1\wedge B_2}(r) &= \frac{4R^2}{5}r - \frac{R}{2}r^2 - \frac{1}{6}r^3 + \frac{1}{8R}r^4 - \frac{1}{240R^3}r^6 \\ f_{B_2/B_1} &= \frac{R}{2}r^2 - \frac{1}{6}r^3 \end{aligned} \tag{44}$$

and when this is inserted in (43) one obtains (5.14). One verifies that $U_{12}(r=0) = \frac{6q^2}{5R}$, the self energy of the sphere, and $U_{12}(r=2R) = \frac{q^2}{2R}$ as it should be.

i) This follows if $P(x) \leq 1$, $0 \leq x \leq 2$ with $P(0) = 0$, $P(2) = 1$. One has $P'(x) = \frac{6}{5} - \frac{3}{2}x^2 + \frac{3}{4}x^3 - \frac{3}{80}x^5$ with $P'(0) = \frac{6}{5}$, $P'(2) = 0$. One notes that $P''(x) = -x\left(3 - \frac{9}{4}x + \frac{3}{16}x^3\right) \leq 0$ in the interval $0 \leq x \leq 2$, implying $P'(x) \geq 0$. Thus $P(x)$ increases from 0 to 1 in this interval and attains its maximum $P(x=2) = 1$.

Ex. 5.4 *Give a heuristic derivation of the lower bound (5.25), assuming that electrons obey Bose statistics.*

The ground state energy of a quantum particle in a cubic box of side L is of the order $\frac{\hbar^2}{m_e L^2}$, thus the kinetic energy of N_e bosons occupying that same state is approximatively $\frac{\hbar^2}{m_e}\frac{N_e}{L^2}$ and (5.22) is replaced by

$$H_{N_e N} \geq k_1 \frac{\hbar^2}{m_e}\frac{N_e}{L^2} - k_2 Z^{2/3} e^2 \frac{N_e^{4/3}}{L} \tag{45}$$

Minimization of (45) gives $L \sim \frac{a_B}{Z^{2/3}N_e^{1/3}}$ which leads to the non extensive bound (5.25)

Ex. 5.5 *Prove the subdomain inequality (5.51) for a system of classical particles.*

One reduces the integration in the total partition function

$$Z(T,\Lambda,N) = \frac{1}{\lambda^{3N}N!}\int_\Lambda \mathrm{d}\mathbf{x}_1 \cdots \int_\Lambda \mathrm{d}\mathbf{x}_N \mathrm{e}^{-\beta V(\mathbf{x}_1,\ldots,\mathbf{x}_N)}$$

to two sub domains Λ_1 and Λ_2. Since the integrand is positive and taking into account that we can have $M, M = 0,1,\ldots,N$, particles in Λ_1 and $N-M$ particles in Λ_2, this gives the inequality

$$\begin{aligned} &Z(T,\Lambda,N) \geq \frac{1}{\lambda^{3N}N!}\sum_{M=0}^{N}\frac{N!}{M!(N-M)!}\int_{\Lambda_1}\mathrm{d}\mathbf{x}_1\cdots\int_{\Lambda_1}\mathrm{d}\mathbf{x}_M \\ &\int_{\Lambda_2}\mathrm{d}\mathbf{x}_{M+1}\cdots\int_{\Lambda_2}\mathrm{d}\mathbf{x}_N \mathrm{e}^{-\beta(V(\mathbf{x}_1,\ldots,\mathbf{x}_M)+V(\mathbf{x}_{M+1},\ldots,\mathbf{x}_N))}\mathrm{e}^{-\beta W(\mathbf{x}_1,\ldots,\mathbf{x}_N)} \end{aligned} \tag{46}$$

The binomial factor corresponds to the number of ways one can chose M particles among N and $W(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ is the inter domain interaction. Keeping only one term $M = N_1, N - M = N_2$ in the sum still strengthens the inequality

$$Z(T,\Lambda,N) \geq \frac{1}{\lambda^{3N_1}N_1!}\int_{\Lambda_1}\mathrm{d}\mathbf{x}_1\cdots\int_{\Lambda_1}\mathrm{d}\mathbf{x}_{N_1}\mathrm{e}^{-\beta(V(\mathbf{x}_1,\ldots,\mathbf{x}_{N_1})}$$
$$\frac{1}{\lambda^{3N_2}N_2!}\int_{\Lambda_2}\mathrm{d}\mathbf{x}_{N_1+1}\cdots\int_{\Lambda_2}\mathrm{d}\mathbf{x}_{N_1+N_2}\mathrm{e}^{-\beta V(\mathbf{x}_{N_1+1},\ldots,\mathbf{x}_{N_1+N_2})}\mathrm{e}^{-\beta W(\mathbf{x}_1,\ldots,\mathbf{x}_{N_1+N_2})} \tag{47}$$

After normalization by the product of the sub domain partition functions, (47) takes the form

$$\frac{Z(T,\Lambda,N)}{Z(T,\Lambda_1,N_1)Z(T,\Lambda_2,N_2)} \geq \left\langle \mathrm{e}^{-\beta W}\right\rangle_{12} \geq \mathrm{e}^{-\beta\langle W\rangle_{12}} \tag{48}$$

where $\langle\cdots\rangle_{12}$ is the canonical statistical average corresponding to the two uncorrelated sub domains Λ_1 and Λ_2. The last inequality in (48) follows from the convexity of the exponential function (Jensen's inequality).

Ex.5.6 *Show (i) that the interaction (5.57) is stable and (ii) that the the sequence of free energies densities (5.62) is bounded below.*

(i) Around a particle at $\mathbf{x}_i$, because of hard cores, there can be only a finite number r_0^3/d^3 of other particles in its interaction range (a sphere of radius r_0). The total interaction potential of this particle is bounded below by $\sum_{j=1}^N V(\mathbf{x}_i - \mathbf{x}_j) \geq -kv_0 r_0^3/d^3$ (k a constant) giving the stability estimate

$$\sum_{i<j=1}^{N} V(\mathbf{x}_i - \mathbf{x}_j) \geq -kv_0\frac{r_0^3}{d^3}\,N \equiv -BN \tag{49}$$

(ii) From the stability (49), one deduces $\mathrm{Tr}_\Lambda \mathrm{e}^{-\beta H} \leq \mathrm{e}^{\beta BN}\mathrm{Tr}_\Lambda \mathrm{e}^{-\frac{1}{2m}\sum_{i=1}^N|\mathbf{p}_i|^2}$ (m the mass of the molecules) and

$$\lim_{\Lambda\to\infty} f_\Lambda(T,\rho) = -\lim_{\Lambda\to\infty}\frac{k_BT}{|\Lambda|}\ln \mathrm{Tr}_\Lambda \mathrm{e}^{-\beta H} \geq \lim_{\Lambda\to\infty} f_{0,\Lambda}(T,\rho) - B\rho = f_0(T,\rho) - B\rho \tag{50}$$

where $f_{0,\Lambda}(T,\rho)$ is the free energy density of a non interacting gas, which is known to have a thermodynamic limit $f_0(T,\rho)$ by explicit calculation.

Exercices of Chapter 6

Ex. 6.1 *Show that in the TL, the average potential $\langle\varphi(\mathbf{r}|\mathbf{0},\mathbf{x}_3,\ldots,\mathbf{x}_N)\rangle^*$ reduces to the expression (6.7).*

The average of the electric potential due to the charges in (6.3) is

$$\left\langle \sum_{j=3}^{N} \frac{e}{|\mathbf{x}_j - \mathbf{r}|} \right\rangle_{\Lambda}^{*} = \int_{\Lambda} \mathrm{d}\mathbf{r}' \frac{e}{|\mathbf{r}' - \mathbf{r}|} \left[\frac{N-2}{Z_{\Lambda}^{*}} \int_{\Lambda^{N-3}} \mathrm{d}\mathbf{x}_4 \cdots \mathrm{d}\mathbf{x}_N \exp(-\beta U_{N-1}(\mathbf{0}, \mathbf{r}', \mathbf{x}_4, \ldots, \mathbf{x}_N)) \right] \tag{51}$$

Comparing the bracket with the definition (6.1) of the two-point correlation $\rho_{N-1}(\mathbf{0}, \mathbf{r}')$ in a system of $N-1$ particles, one sees that it differs from $\rho_{N-1}(\mathbf{0}, \mathbf{r}')$ by the factor

$$\frac{1}{N-1} \frac{\int_{\Lambda^{N-1}} \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{x}_3 \cdots \mathrm{d}\mathbf{x}_N \exp(-\beta U_{N-1}(\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_N))}{\int_{\Lambda^{N-2}} \mathrm{d}\mathbf{x}_3 \cdots \mathrm{d}\mathbf{x}_N \exp(-\beta U_{N-1}(\mathbf{0}, \mathbf{x}_3, \ldots, \mathbf{x}_N))} \tag{52}$$

In view of the translational invariance of $U_{N-1}(\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_N)$ the ratio of these two configurational integrals behaves as $|\Lambda|, |\Lambda| \to \infty$, so that the factor (52) tends to $1/\rho$. Hence

$$\lim_{N\to\infty, N/|\Lambda|=\rho} \left\langle \sum_{j=3}^{N} \frac{e}{|\mathbf{x}_j - \mathbf{r}|} \right\rangle^{*} = \frac{1}{\rho} \int \mathrm{d}\mathbf{r}' \frac{e}{|\mathbf{r}' - \mathbf{r}|} \rho(\mathbf{0}, \mathbf{r}') \tag{53}$$

Ex. 6.2 *Show that in the TL, the normalization factor $A_{N,\Lambda}$ defined by (6.9) goes to ρ^2.*

For Λ large, the denominator of $A_{N,\Lambda}$, with the decomposition (6.2) and the approximation (6.6), behaves like

$$\begin{aligned} &|\Lambda| \int_{\Lambda^{N-2}} \mathrm{d}\mathbf{x}_3 \cdots \mathrm{d}\mathbf{x}_N \left[\int_{\Lambda} \mathrm{d}\mathbf{r} \mathrm{e}^{-\beta\varphi(\mathbf{r}|\mathbf{0}, \mathbf{x}_3, \ldots, \mathbf{x}_N)} \right] \mathrm{e}^{-\beta U_{N-1}(\mathbf{0}, \mathbf{x}_3, \ldots, \mathbf{x}_N)} \\ &= |\Lambda| Z_{\Lambda}^{*} \int_{\Lambda} \mathrm{d}\mathbf{r} \langle \exp(-\beta e \varphi(\mathbf{r}|\mathbf{0}, \mathbf{x}_3, \ldots, \mathbf{x}_N) \rangle_{\Lambda}^{*} \sim |\Lambda| Z_{\Lambda}^{*} \int_{\Lambda} \mathrm{d}\mathbf{r} \mathrm{e}^{-\beta\varphi(\mathbf{r})} \end{aligned} \tag{54}$$

The volume factor $|\Lambda|$ stems from the translational invariance of $U(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N))$, and $\int_{\Lambda} \mathrm{d}\mathbf{r} \mathrm{e}^{-\beta\varphi(\mathbf{r})} \sim |\Lambda|$, since $\varphi(\mathbf{r})$ (6.7) vanishes as $|\mathbf{r}| \to \infty$. Hence $A_{N,\Lambda} \sim \frac{N(N-1)}{|\Lambda|^2} \to \rho^2$.

Ex. 6.3 *Extend the heuristic derivation of Debye theory for the OCP to a TCP made of charged hard spheres of diameter σ. Show that the validity of this theory requires the double condition $\beta e^2/a \ll 1$ and $\rho\sigma^3 \ll 1$.*

The solution to this exercise will be completed as soon as possible.

Ex. 6.4 *Determine the symmetry factors of the diagrams Figures 6.1 and 6.2.*

Analyse the figures 6.1 and 6.2.

Ex. 6.5 *Derive formula (6.44) by starting from (6.43).*

The Fourier transform of $H_N(\alpha_\mathrm{a}, \alpha_\mathrm{b}, \mathbf{x})$

$$\tilde{H}_N(\alpha_\mathrm{a}, \alpha_\mathrm{b}, \mathbf{k}) = \sum_{\alpha_1 \ldots \alpha_N} \rho_{\alpha_1} \tilde{b}_C(\alpha_a, \alpha_1, \mathbf{k}) \rho_{\alpha_2} \tilde{b}_C(\alpha_1, \alpha_2, \mathbf{k}) \cdots \rho_{\alpha_N} \tilde{b}_C(\alpha_N, \alpha_b, \mathbf{k}) \tag{55}$$

is the product of $N+1$ bonds $\tilde{b}_C(\alpha_i, \alpha_j, \mathbf{k}) = -\beta \frac{e_{\alpha_i} e_{\alpha_j}}{|\mathbf{k}|^2}$,

$$\tilde{H}_N(\alpha_\mathrm{a}, \alpha_\mathrm{b}, \mathbf{k}) = -\frac{4\pi\beta e_{\alpha_a}}{k^2} \left(\frac{-\beta}{k^2}\right)^N \left[\sum_{\alpha_1 \ldots \alpha_N} \prod_{i=1}^{N} 4\pi\beta \rho_{\alpha_i} e_{\alpha_i}\right] e_{\alpha_b} \tag{56}$$

which yields (6.44) and (6.45).

Ex. 6.6 *Derive the expression (6.48) for the Abe-Meeron bond.*

Set $\beta_{ij} = \beta \mathrm{e}_{\gamma_i} \mathrm{e}_{\gamma_j}$,

$$\begin{aligned} &\exp(-\beta_{ij}(\phi_\mathrm{D} - v_\mathrm{C})) - 1 + \beta_{ij}(\phi_\mathrm{D} - v_\mathrm{C}) \\ &\quad + [\exp(-\beta V_\mathrm{sr} - \beta_{ij} v_\mathrm{C}) - 1 + \beta_{ij} v_\mathrm{C}))] \exp(-\beta_{ij}(\phi_\mathrm{D} - v_\mathrm{C})) \\ &\quad - \beta_{ij} v_\mathrm{C} \left[\exp(-\beta_{ij}(\phi_\mathrm{D} - v_\mathrm{C})) - 1\right] \\ &= \exp\left[-\beta V_\mathrm{sr} - \beta_{ij}\phi_\mathrm{D}\right] - 1 + \beta_{ij}\phi_\mathrm{D} = b_\mathrm{AM} \end{aligned} \tag{57}$$

Exercices of Chapter 7

Ex. 7.1 *Apply the theorem of residues for calculating the long-distance behavior of the integral over q in the $\Gamma^{3/2}$-term of expansion (7.12).*

The solution to this exercise will be completed as soon as possible.

Ex. 7.2 *Derive the leading behavior (7.24) in the zero-density limit of the constant (7.23).*

The solution to this exercise will be completed as soon as possible.

Ex. 7.3 *Show that $\boldsymbol{\nabla}_\mathrm{b} A_\mathrm{conv}(\mathbf{r}_\mathrm{b})$ and $\boldsymbol{\nabla}_\mathrm{b} B(\mathbf{r}_\mathrm{b})$ are finite and vanish at $\mathbf{r}_\mathrm{b} = \mathbf{0}$ by using (i) the behavior (7.27) in $A_\mathrm{conv}(\mathbf{r}_\mathrm{b})$ (ii) a suitable reorganization of the screened Mayer series for $B(\mathbf{r}_\mathrm{b})$ in terms of the full bond $(b_\mathrm{D} + b_\mathrm{AM})$.*

The solution to this exercise will be completed as soon as possible.

Ex. 7.4 *Derive (7.50) and 7.51) for the RPM.*

The solution to this exercise will be completed as soon as possible.

Ex.7.5 *Solve (7.63) perturbatively at low densities and recover the lowest-order correction to ℓ_D given by (7.16).*

The solution to this exercise will be completed as soon as possible.

Ex. 7.6 *Solve (7.64) numerically at finite values of* ϵ.

The solution to this exercise will be completed as soon as possible.

Ex. 7.7 *Derive formula (7.68).*

The solution to this exercise will be completed as soon as possible.

Ex. 7.8 *Calculate the inverse Fourier transform of (7.73). Determine the constant in (7.74) for* $\kappa_{\mathrm{D}} d < 1/2$, *and derive the corresponding expression of* $\phi_{\mathrm{DS}}(r)$ *for* $\kappa_{\mathrm{D}} d = 1/2$ *and* $\kappa_{\mathrm{D}} d > 1/2$.

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 8

Ex. 8.1 *Show that the Sine-Gordon action (8.10) converges to its Debye-Hückel form (8.17) as* $\beta \to 0$.

Introducing $\kappa_z = (8\pi\beta e^2 z)^{1/2}$, a limited Taylor expansion in (8.10) leads to

$$\begin{aligned} 2\bar{z} &= 2z + \frac{\kappa_z^2}{8\pi d} + \mathrm{O}\left(\frac{\kappa_z^2 \beta}{d^2}\right), \\ \int_\Lambda \mathrm{d}\mathbf{x} \cos\sqrt{\beta} e\phi(\mathbf{x}) &= |\Lambda| - \frac{1}{2}\beta e^2 \int_\Lambda \mathrm{d}\mathbf{x}\phi^2(\mathbf{x}) + \mathrm{O}(\beta^2) \end{aligned} \tag{58}$$

Since κ_z remains finite in the limit $\beta \to 0$, the terms $\frac{\beta}{d} \sim \beta^{1/3-s}$ and $\frac{\beta}{d^2} \sim \beta^{2/3-2s}$ vanish if $s < \frac{1}{6}$, see (8.4). Thus the difference

$$2\bar{z}\int_\Lambda \mathrm{d}\mathbf{x} \cos\sqrt{\beta} e\phi(\mathbf{x}) - \left(2z + \frac{\kappa_z^2}{8\pi d}\right)|\Lambda| - \frac{\kappa_z^2}{8\pi}\int_\Lambda \mathrm{d}\mathbf{x}\phi^2(\mathbf{x}) \tag{59}$$

vanishes as $\beta \to 0$.

Ex. 8.2 *Derive the pressure from the Debye-Hückel action (8.17).*

Setting $\Xi_{\mathrm{ideal}} = \exp(2|\Lambda|z)$, the contribution of the ideal TCP gas, one has

$$\frac{\Xi_{\mathrm{DH},\Lambda}}{\Xi_{\mathrm{ideal},\Lambda}} = \int \mathrm{d}\mathbf{x}\phi(\cdot)\mathrm{e}^{-S_{\mathrm{DH}}\{\phi(\cdot)\}} = \left\langle \exp\left(-\frac{\kappa_z^2}{8\pi}\int_\Lambda \mathrm{d}\mathbf{x}\phi^2(\mathbf{x})\right)\right\rangle_{v^{-1}} \exp\left(|\Lambda|\frac{\kappa_z^2}{8\pi d}\right). \tag{60}$$

The Gaussian average can be calculated with the help of the formula (C.6) with $A = v^{-1}$ and $B = \frac{\kappa_z^2}{4\pi} I$, namely

$$\left\langle \exp\left(-\frac{\kappa_z^2}{8\pi}\int_\Lambda \mathrm{d}\mathbf{x}\phi^2(\mathbf{x})\right)\right\rangle_{v^{-1}} = \left[\det\left(I + \frac{\kappa_z^2}{4\pi}v\right)\right]^{-1/2}$$
$$= \exp\left[-\frac{1}{2}\mathrm{Tr}_\Lambda \ln\left(I + \frac{\kappa_z^2}{4\pi}v\right)\right] = \exp\left[-\frac{1}{2}\sum_{\mathbf{k}_n}\ln(1 + \frac{\kappa_z^2}{4\pi}v(\mathbf{k}_n))\right]. \quad (61)$$

In order to obtain the second line, we have used the metallic walls boundary conditions for a box Λ of volume L^3 and written v in diagonal form. The sum runs on all the eigenmodes $\mathbf{k}_n = \{k_n = \frac{\pi n}{L}\}$ of the box Λ and (see Appendix E, Section 3.2)

$$\sum_{\mathbf{k}_n}\tilde{v}(\mathbf{k}_n) \sim \frac{L^3}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\tilde{v}(\mathbf{k}) = L^3 v(0) = \frac{L^3}{d} \quad (62)$$

As $L \to \infty$, (60) becomes with (61) and (62)

$$\frac{\Xi_{\mathrm{DH},\Lambda}}{\Xi_{\mathrm{ideal},\Lambda}} \sim \exp\left[-\frac{L^3}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\left[\ln\left(1 + \frac{\kappa_z^2}{4\pi}v(\mathbf{k})\right) - \frac{\kappa_z^2}{4\pi}v(\mathbf{k})\right]\right]$$
$$\sim \exp\left[-\frac{L^3}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\left[\ln\left(1 + \frac{\kappa_z^2}{k^2}\right) - \frac{\kappa_z^2}{k^2}\right]\right] = \exp\left(L^3\frac{\kappa_z^3}{12\pi}\right), \quad d \to 0 \quad (63)$$

In the second line we remove the short distance cut-off $d \to 0$, and the last equality follows from an direct calculation of the $\mathbf{k}$-integral using

$$\int \mathrm{d}x x^2 \ln(1 + \frac{1}{x^2}) = \frac{x^3}{3}\ln\left(1 + \frac{1}{x^2}\right) + \frac{2x}{3} - \frac{2}{3}\mathrm{arctg}x.$$

This gives the grand canonical Debye-Hückel pressure

$$P_{\mathrm{DH}} = \lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\ln \Xi_{\mathrm{DH},\Lambda} = 2z + \frac{\kappa_z^3}{12\pi} \quad (64)$$

Ex. 8.3 *The dipole correlations decay as* r^{-3}.

In the Sine-Gordon representation, the two point dipole distribution is

$$\rho(\mathbf{r}_1, \mathbf{p}_1; \mathbf{r}_2, \mathbf{p}_2) = \bar{z}^2\left\langle \mathrm{e}^{i\sqrt{\beta}\mathbf{p}_1\cdot\boldsymbol{\nabla}_1\phi(\mathbf{r}_1)}\mathrm{e}^{i\sqrt{\beta}\mathbf{p}_2\cdot\boldsymbol{\nabla}_2\phi(\mathbf{r}_2)}\right\rangle \quad (65)$$

where the average is defined with the Sine-Gordon action (8.40). In the Gaussian approximation, the average $\langle \mathrm{e}^{i\sqrt{\beta}\mathbf{p}_1\cdot\boldsymbol{\nabla}_1\phi(\mathbf{r}_1)}\mathrm{e}^{i\sqrt{\beta}\mathbf{p}_2\cdot\boldsymbol{\nabla}_1\phi(\mathbf{r}_2)}\rangle_{\mathrm{mf}}$ can easily be calculated from the basic formula (C.10) and it takes the form

$$\rho_{\mathrm{mf}}(\mathbf{r}_1, \mathbf{p}_1; \mathbf{r}_2, \mathbf{p}_2) = \rho_{\mathrm{mf}}(\mathbf{r}_1, \mathbf{p}_1)\rho_{\mathrm{mf}}(\mathbf{r}_2, \mathbf{p}_2))\mathrm{e}^{-\beta(\mathbf{p}_1\cdot\boldsymbol{\nabla}_1)(\mathbf{p}_2\cdot\boldsymbol{\nabla}_2)v^{\mathrm{eff}}_{\mathrm{dip,eff}}(\mathbf{r}_1-\mathbf{r}_2)} \quad (66)$$

It follows from (8.45) that $(\mathbf{p}_1\cdot\boldsymbol{\nabla}_1)(\mathbf{p}_2\cdot\boldsymbol{\nabla}_2)v^{\mathrm{mf}}_{\mathrm{dip,eff}}(\mathbf{r}_1-\mathbf{r}_2)=(\mathbf{p}_1\cdot\boldsymbol{\nabla}_1)(\mathbf{p}_2\cdot\boldsymbol{\nabla}_2)\frac{1}{\varepsilon|\mathbf{r}_1-\mathbf{r}_2|}$ decays as $|\mathbf{r}_1-\mathbf{r}_2|^{-3}$.

Ex. 8.4 *The correlations of a mixture of dipole and charges decay exponentially fast.*

The microscopic density of ions is $\sum_{i=1}^{N}(e_{\alpha_i}+\mathbf{p}_{\alpha_i}\cdot\boldsymbol{\nabla}_i)\delta(\mathbf{x}-\mathbf{x}_i)$ with corresponding action $S\{\phi(\cdot)\}=\frac{1}{2}(\phi,v^{-1}\phi)-\sum_{\alpha=1}^{\mathcal{S}}\int \mathrm{d}\mathbf{x}z_\alpha(\mathbf{x})\exp[i\sqrt{\beta}(e_\alpha+\mathbf{p}_\alpha\cdot\boldsymbol{\nabla})\phi(\mathbf{x})]$ The contribution of the charges to the quadratic term is the same as in the Debye-Hückel action (8.17). Thus, the effective potential in the Gaussian approximation, which is found to be $4\pi\left[(1+\frac{4\pi}{3}\sum_\alpha z_\alpha|\mathbf{p}_\alpha|^2)k^2+\kappa_z^2+8\pi i\sum_\alpha z_\alpha\mathbf{p}_\alpha\cdot\mathbf{k}\right]^{-1}$, is screened.

Ex. 8.5 *Establish the YBG Equation (8.49).*

Substitute the first BGY equation (8.46)

$$\beta^{-1}\boldsymbol{\nabla}\rho(\mathbf{r})=e_\gamma\mathbf{E}(\mathbf{r})+\int \mathrm{d}\mathbf{r}'\mathbf{F}(\mathbf{r},\mathbf{r}')[\rho(\mathbf{r}',\mathbf{r})-\rho(\mathbf{r}')\rho(\mathbf{r})]$$

in the r.h.s of (F.9)

$$\beta^{-1}\boldsymbol{\nabla}\rho(\mathbf{r},\mathbf{r}_1,\ldots,\mathbf{r}_n)-\beta^{-1}\boldsymbol{\nabla}\rho(\mathbf{r})\rho(\mathbf{r}_1,\ldots,\mathbf{r}_n).$$

From the definition of the excess charge density one finds :

$$\rho(\mathbf{r})e_\gamma\mathbf{E}_{\mathrm{exc}}(\mathbf{r}|\mathbf{r}_1,\ldots,\mathbf{r}_n)$$

$$=\int \mathrm{d}\mathbf{r}'\mathbf{F}(\mathbf{r},\mathbf{r}')\left[\rho(\mathbf{r})\rho(\mathbf{r}',\mathbf{r}_1,\ldots,\mathbf{r}_n)-\rho(\mathbf{r})\rho(\mathbf{r}')\rho(\mathbf{r}_1,\ldots,\mathbf{r}_n)\right]$$

$$+\sum_{i=1}^{n}\mathbf{F}(\mathbf{r},\mathbf{r}_i)\rho(\mathbf{r})\rho(\mathbf{r}_1,\ldots,\mathbf{r}_n).$$

Introducing the expression (B.12) of the truncated function $\rho(\mathbf{r}'|\mathbf{r}|\mathbf{r}_1,\ldots,\mathbf{r}_n)$ shows the equivalence of (8.49) and (8.46).

Ex. 8.6 *Recover the Stillinger-Lovett sum rule from (8.68).*

With $s=1$, $b=1$, $\Gamma(1)=1$, $\Gamma(1/2)=\sqrt{\pi}$, one has $\tilde{\rho}^{(2)}(\mathbf{k})+\rho=\frac{1}{4\pi\beta}|\mathbf{k}|^2$, $|\mathbf{k}|\to 0$, which is the Stillinger-Lovet rule for the jellium (with $e=1$), see (2.16).

Ex. 8.7 *Show that the correlations of a two dimensional Coulomb fluid with potential $1/r$ (see Section 4.1.4) decay like $1/r^3$.*

This two dimensional fluid in the plane $\left\{r^1,r^2\right\}$ has the long range potential $\frac{1}{r}$, $r=\sqrt{(r^1)^2+(r^2)^2}$. In two dimensions the formula (8.69) is modified to $\rho^{(2)}(\mathbf{r})\sim\frac{1}{\beta c_s c_{4-s}}\frac{1}{r^{4-s}}$, $\quad c_s=2^{2-s}\pi\frac{\Gamma((2-s)/2)}{\Gamma(s/2)}$. For $s=1$, with $\Gamma(-1/2)=-2\sqrt{\pi}$, $\Gamma(3/2)=\frac{\sqrt{\pi}}{2}$, $b=1$, one finds $\rho^{(2)}(\mathbf{r})\sim-\frac{1}{4\pi^2\beta r^3}$. This is in agreement with the decay of the correlations in a slab Section 2.4.1.1

Ex. 8.8 *Derive the following properties:*

(i) A uniform and isotropic state obeying (F.9) with $c_{\rm ext}(\mathbf{r}) = 0$*, behaves necessarily as a conductor at the macroscopic level.*

(ii) If the sample is subjected to a homogeneous external electric field $\mathbf{E}_{\rm ext}$*, the charge density is concentrated inside boundary layers.*

(i) In a translation and rotation invariant state, the densities ρ_γ and the electric field $\mathbf{E}$ are uniform so that the first equation (F.9) becomes

$$0 = e_{\gamma_1}\rho_{\gamma_1}\mathbf{E} + \sum_\gamma \int \mathrm{d}\mathbf{r}\mathbf{F}(\mathbf{r},\mathbf{r}_1)[\rho(\mathbf{r},\mathbf{r}_1) - \rho_{\gamma_1}\rho_\gamma] \tag{67}$$

and one can set $\mathbf{r}_1 = 0$. The two point correlation $\rho(\mathbf{r}, 0)$ depends only on the distance $|\mathbf{r}|$ whereas the force $\mathbf{F}(\mathbf{r}, 0)$ is an odd function of $\mathbf{r}$, thus the integral vanishes. One concludes that an homogeneous isotropic state satisfying the condition (F.10) cannot sustain a non vanishing electric field in its bulk and thus corresponds to a perfect conductor in the sense of macroscopic electrostatics. As a consequence, the charge density $c = \sum_\gamma e_\gamma\rho_\gamma$ vanishes in the bulk .

(ii) Coming back to the finite volume expression (F.7) of the total electric field, we conclude that at any point $\mathbf{r}$ in the bulk

$$\mathbf{E}_\Lambda(\mathbf{r}) = -\boldsymbol{\nabla}_\mathbf{r} \int_\Lambda \mathrm{d}\mathbf{r}'\ v(\mathbf{r}-\mathbf{r}')c_{\mathbf{sys}}(\mathbf{r}') + \mathbf{E}_{\rm ext} = 0 \tag{68}$$

Because of the local neutrality $c_{\mathbf{sys}}(\mathbf{r}) = 0$ when $\mathbf{r}$ is in the bulk, we see that any nonvanishing part of charge density in the finite volume system has to be concentrated near the boundary of Λ. According to (68), these boundary layers generate an electric field that exactly compensates the external field in the interior of the sample.

Exercices of Chapter 9

Ex. 9.1 *Derive the total potential energy relation (9.3) and determine the constants* $\varphi(0)$ *and* C*.*

The total potential energy is

$$U(\mathbf{x}_1,\ldots,\mathbf{x}_N) = -e^2\sum_{i<j}^N \ln\frac{|\mathbf{x}_i-\mathbf{x}_j|}{L} + e\sum_{i=1}^N \varphi_{\rm B}(\mathbf{x}_i) - \frac{e\rho}{2}\int_{r\le R}\mathrm{d}\mathbf{x}\varphi_{\rm B}(\mathbf{x}), \quad r = |\mathbf{x}| \tag{69}$$

where $\varphi_{\rm B}(\mathbf{x})$ is solution of the radial Poisson equation in two dimensions

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_{\rm B}}{\partial r}\right) = -2\pi(-e\rho), \quad r \le R \tag{70}$$

The two last terms in the r.h.s of (69) are the energy of the particles in the background and the self energy of the background. The solution of (70) is equal to (9.2) with

$$\varphi_B(0) = e\rho \int_{r\le R} \mathrm{d}^2 r \ln(\tfrac{r}{L}) = Ne\left(\ln R - \ln L - \frac{1}{2}\right), \quad N = \rho\pi R^2$$

and the background energy is

$$-\frac{e\rho}{2}\int_{r\le R} \mathrm{d}^2\mathbf{r}\varphi_{\mathrm{B}}(r) = -\frac{N^2e^2}{2} - \frac{Ne}{2}\varphi_{\mathrm{B}}(0).$$

Collecting all the terms independent of the particle coordinates leads to

$$C = \frac{N^2e^2}{2}\left(\ln R - \frac{3}{4}\right) - \frac{Ne^2}{2}\ln L$$

Ex. 9.2 *Check that the pair correlation (9.13) satisfies the Stillinger-Lovett second moment condition.*

In the OCP the structure function is given by (2.32). Thus $\int \mathrm{d}^2\mathbf{r}S(\mathbf{r}) = -e^2\rho^2\int \mathrm{d}^2\mathrm{e}^{-\pi\rho|\mathbf{r}|^2} + e^2\rho = 0$ and $(1/4)\int \mathrm{d}^2|\mathbf{r}|^2 S(\mathbf{r}) = (e^2\rho^2/4)\int \mathrm{d}^2|\mathbf{r}|^2\mathrm{e}^{-\pi\rho|\mathbf{r}|^2} = \mathrm{e}^2/4\pi$.

Ex. 9.3 *Derive the Cauchy double alternant formula.*

Establish the formula (9.15) for $N = 2$, a general proof can be found in [3], p.142.

Ex. 9.4 *Establish the determinant formula (9.20)*

Note that the terms in the series (9.19) can be expressed as determinants of antihermitian matrices, i.e.

$$|\det A|^2 = \det\begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}, \qquad A_{pq} = \frac{L}{u_p - v_q} \tag{71}$$

where A^* is the hermitian conjugate of $A, A^*_{pq} = \bar{A}_{qp}$. Then expand the determinant

$$\det\begin{pmatrix} I_M & f_+A_M \\ -f_-A^*_M & I_M \end{pmatrix} \tag{72}$$

in powers of f_+f_-.

Ex. 9.5 *Derive the expressions (9.30) and (9.31) for the density and the correlations.*

Use the formula $\ln\det A = \mathrm{Tr}\ln A$ in (9.28)

$$\ln\Xi = \mathrm{Tr}\left[\ln B - \ln(\sigma_x\partial_x + \sigma_y\partial_y)\right], \quad B = (\sigma_x\partial_x + \sigma_y\partial_y + f_+(\mathbf{r})P_+ + f_-(\mathbf{r})P_-) \tag{73}$$

Then one has

$$\frac{\delta \ln \Xi}{\delta f_{s_1}(\mathbf{r}_1)} = \mathrm{Tr}\left[B^{-1}\frac{\delta B}{\delta f_{s_1}(\mathbf{r}_1)}\right] \tag{74}$$

and when calculating the trace by means of a complete set of vectors $|\mathbf{r}, s\rangle$ ($\int \mathrm{d}\mathbf{r} \sum_s |\mathbf{r}, s\rangle\langle\mathbf{r}, s| = I$), the projector

$$\frac{\delta B}{\delta f_{s_1}(\mathbf{r}_1)} = \delta(\mathbf{r}_1 - \mathbf{r})P_{s_1} \tag{75}$$

selects the values $\mathbf{r}_1$ and s_1, hence the result (9.30). In the same way for the two point truncated function

$$\begin{aligned}\frac{\delta^2 \ln \Xi}{\delta f_{s_2}(\mathbf{r}_2)\delta f_{s_1}(\mathbf{r}_1)} &= \mathrm{Tr}\left[\left(\frac{\delta B^{-1}}{\delta f_{s_2}(\mathbf{r}_2)}\right)\frac{\delta B}{\delta f_{s_1}(\mathbf{r}_1)}\right] \\ &= -\mathrm{Tr}\left[B^{-1}\left(\frac{\delta B}{\delta f_{s_2}(\mathbf{r}_2)}\right)B^{-1}\frac{\delta B}{\delta f_{s_1}(\mathbf{r}_1)}\right]\end{aligned} \tag{76}$$

Using (75) and evaluating the trace with the set of vectors $|\mathbf{r}, s\rangle$ leads to (9.31).

Ex. 9.6 *Consider the expression (7.50) for the charge correlations of the 2D version of the RPM, and determine its poles in the complex plane by keeping only the first Abe-Meeron diagram for the short-range parts of the direct correlation functions. Show that the corresponding inverse screening length ℓ_{S}^{-1} behaves, in the low-density limit, like the expression (9.39) for m, except for multiplicative constants.*

The solution to this exercise will be completed as soon as possible.

Ex. 9.7 *Derive expansion (9.52) by applying the method introduced in Section 7.4.1 for the 3D case: determine the poles of the expression (7.50) in the complex plane where the short-range parts of the direct correlations are replaced by their first Abe-Meeron diagrams.*

The solution to this exercise will be completed as soon as possible.

Ex. 9.8 *Within the mean-field approximation, calculate $\psi_{+}^{(\mathrm{av})}(r)$ solution of (9.57)* and (9.58).

The solution to this exercise will be completed as soon as possible.

Ex. 9.9 *Derive the STLS integral equation in Fourier space for the static structure factor $\tilde{S}(\mathbf{k})$ of the 3D OCP. Show that in 2D, this equation becomes purely local.*

The solution to this exercise will be completed as soon as possible.

Ex. 9.10 *Within the GDH approximation applied to the RPM, show that $\tilde{S}(\mathbf{k})$ and $\tilde{N}(\mathbf{k})$ reduce to simple expressions in terms of elementary functions.*

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 10

Ex. 10.1 *Derive expression (10.28) treating the surrounding plasma in the Debye-Hückel approximation.*

The solution to this exercise will be completed as soon as possible.

Ex. 10.2 *Establish the asymptotic formula (10.32).*

Working out $(W(\mathcal{F}_a,\mathcal{F}_b))^2$ from (10.11) at lowest order in $\hbar$ gives (with summation on repeated cartesian indices μ,ν,ϵ,η)

$$\begin{aligned}&(W(\mathcal{F}_a,\mathcal{F}_b))^2 \sim\\ &e_a^2e_b^2\lambda_a^2\lambda_b^2\int_0^1 \mathrm{d}s_1\int_0^1 \mathrm{d}s_2(\delta(s_1-s_2)-1)\int_0^1 \mathrm{d}u_1\int_0^1 \mathrm{d}u_2(\delta(u_1-u_2)-1)\\ &\qquad \xi_a^\mu(s_1)\xi_b^\nu(s_2)\xi_a^\epsilon(u_1)\xi_b^\eta(u_2)[\partial_{r_a}^\mu\partial_{r_b}^\nu v(\mathbf{r}_a-\mathbf{r}_b)][\partial_{r_a}^\epsilon\partial_{r_b}^\eta v(\mathbf{r}_a-\mathbf{r}_b)]\rho_a\rho_b \quad (56)\end{aligned}$$

Since the prefactor $\lambda_a^2\lambda_b^2$ is already of order $\hbar^4$, one has replaced $\bar{D}(\boldsymbol{\xi})$ by $D(\boldsymbol{\xi})$, dropping the $\boldsymbol{\xi}$ dependance in (10.26). The Gaussian averages on the filaments $\boldsymbol{\xi}_a$ and $\boldsymbol{\xi}_b$ is calculated with the help of the covariance (G.7) with the result

$$\begin{aligned}&\delta_{\mu\epsilon}\delta_{\nu\eta}\int_0^1 \mathrm{d}s_1\int_0^1 \mathrm{d}s_2\int_0^1 \mathrm{d}u_1\int_0^1 \mathrm{d}u_2(\delta(s_1-s_2)-1)(\delta(u_1-u_2)-1)\\ &\times(\min(s_1u_1)-s_1u_1)(\min(s_2u_2)-s_2u_2)=\frac{1}{720}\delta_{\mu\epsilon}\delta_{\nu\eta} \qquad (77)\end{aligned}$$

and

$$\delta_{\mu\epsilon}\delta_{\nu\eta}[\partial_{r_a}^\mu\partial_{r_b}^\nu v(\mathbf{r}_a-\mathbf{r}_b)][\partial_{r_a}^\epsilon\partial_{r_b}^\eta v(\mathbf{r}_a-\mathbf{r}_b)]\sim\frac{6}{|\mathbf{r}_a-\mathbf{r}_b|^6},\quad |\mathbf{r}_a-\mathbf{r}_b|\to\infty \quad (78)$$

Inserting (77), (78) in (56) and (10.30) leads to (10.32).

Ex. 10.3 *Derive the $\hbar^2$-correction (10.60) for the OCP considered as the limit of a TCP described above.*

The solution to this exercise will be completed as soon as possible.

Ex. 10.4 *Show that the neutrality for a multi-component system is preserved at order $\hbar^2$.*

One use translation invariance to write in the first term of the r.h.s of (10.61) , spelling out the particle indices,

$$\begin{aligned}&\rho_{\rm cl}(\alpha\mathbf{r}_a,\gamma\mathbf{r},\gamma'\mathbf{r}')=\rho_{\rm cl}(\alpha,\mathbf{r}_a-\mathbf{r}',\gamma,\mathbf{r}-\mathbf{r}',\gamma',0)\\ &\rho_{\rm cl}(\gamma\mathbf{r},\gamma'\mathbf{r}')=\rho_{\rm cl}(\gamma,\mathbf{r}-\mathbf{r}',\gamma',0)\qquad \Delta_{\mathbf{r}}V(\mathbf{r},\mathbf{r}')=\Delta_{\mathbf{r}}V(\mathbf{r}-\mathbf{r}') \qquad (79)\end{aligned}$$

Shift the $\mathbf{r}$ integral into $\mathbf{r}+\mathbf{r}'$ and then shfft the $\mathbf{r}_a$ integral into $\mathbf{r}_a+\mathbf{r}'$. After summation on charges e_α, the r.h.s of (10.61) becomes

$$\begin{aligned}&-\frac{\beta}{4\pi}\sum_\gamma\sum_{\gamma'}\lambda_\gamma^2\int \mathrm{d}\mathbf{r}\Delta_{\mathbf{r}}V(\mathbf{r})\sum_\alpha e_\alpha\int \mathrm{d}\mathbf{r}_a[\rho_{\rm cl}(\alpha\mathbf{r}_a,\gamma\mathbf{r},\gamma',0)-\rho_{\rm cl}(\gamma\mathbf{r},\gamma'0)]\\ &\qquad\qquad=\frac{\beta}{4\pi}\sum_\gamma\sum_{\gamma'}\lambda_\gamma^2\int \mathrm{d}\mathbf{r}\Delta_{\mathbf{r}}V(\mathbf{r})[e_\gamma+e_{\gamma'}]\rho_{\rm cl}(\gamma\mathbf{r},\gamma'0) \quad (80)\end{aligned}$$

where we have applied the charge sum rule (2.50). Using again translation invariance, the second term of the r.h.s of (10.61) reads, after summation on the charges e_α and setting $\alpha = \gamma'$

$$-\frac{\beta}{4\pi}\sum_{\gamma}\sum_{\gamma'}\lambda_\gamma e_{\gamma'}\int \mathrm{d}\mathbf{r}\Delta_{\mathbf{r}}V(\mathbf{r})\rho(\gamma'\mathbf{r},\gamma 0) - \frac{\beta}{4\pi}\sum_{\gamma}\sum_{\gamma'}[\lambda_{\gamma'} e_{\gamma'}\int \mathrm{d}\mathbf{r}\Delta_{\mathbf{r}}V(\mathbf{r})\rho(\gamma'\mathbf{r},\gamma 0)$$
$$-\frac{\beta}{4\pi}\sum_{\gamma}\sum_{\gamma'}\lambda_\gamma e_{\gamma'}\int \mathrm{d}\mathbf{r}\Delta_{\mathbf{r}}V(\mathbf{r})\rho(\gamma\mathbf{r},\gamma' 0) - \frac{\beta}{4\pi}\sum_{\gamma}\sum_{\gamma'}\lambda_{\gamma} e_{\gamma}\int \mathrm{d}\mathbf{r}\Delta_{\mathbf{r}}V(\mathbf{r})\rho(\gamma\mathbf{r},\gamma' 0) \tag{81}$$

In the first term we use the symmetries $\rho(\gamma'\mathbf{r},\gamma 0) = \rho(\gamma 0,\gamma'\mathbf{r}) = \rho(\gamma,-\mathbf{r},\gamma' 0) = \rho(\gamma\mathbf{r},\gamma' 0)$ and interchange the indices γ,γ' in the second term. This cancels the expression (80).

Ex. 10.5 *Show that the $\hbar^2$-correction for the pair correlations of a multi-component system vanishes exponentially fast at large distances.*

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 11

Ex.11.1 *Determine the number of permutations of $N = 6$ particle positions which lead to three loops made with respectively one, two and three particles (Figure 11.2)*

The solution to this exercise will be completed as soon as possible.

Ex. 11.2 *Check that the first terms of Ξ_Λ up to three particles are entirely and exactly provided by the contributions of the magic formula (11.6) involving at most three loops.*

The solution to this exercise will be completed as soon as possible.

Ex. 11.3 *Retrieve from formula (11.6) the pressure of ideal Bose or Fermi gases.*

If there are no interactions, $\Xi_\Lambda = \mathrm{e}^{\int_\Lambda D(\mathcal{L})z(\mathcal{L})}$. Since the $D(\boldsymbol{\mathcal{X}})$ measure is normalized to 1, one has with (H.19) and (H.18)

$$\int_\Lambda D(\mathcal{L})z(\mathcal{L}) = 2|\Lambda|\sum_{q=1}^{\infty}\frac{(\eta)^{q-1}}{q}\frac{\mathrm{e}^{\beta\mu q}}{(2\pi q\lambda^2)^{3/2}} = \frac{2|\Lambda|}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\sum_{q=1}^{\infty}\frac{(\eta)^{q-1}}{q}\mathrm{e}^{-\beta q[\epsilon(\mathbf{k})-\mu]} \tag{82}$$

with $\epsilon(\mathbf{k}) = \frac{\hbar^2k^2}{2m}$. The q series sum up to $\ln\left(1-\eta\mathrm{e}^{-\beta[\epsilon(\mathbf{k})-\mu]}\right)$. When this is inserted in (82) one finds

$$P_0(\mu) = \lim_{|\Lambda|\to\infty}\frac{k_BT}{|\Lambda|}\ln\Xi_\Lambda = \frac{2k_BT}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\ln\left(1-\eta\mathrm{e}^{-\beta[\epsilon(\mathbf{k})-\mu]}\right) \tag{83}$$

which is the standard expression of the pressure of a Bose ($\eta = 1$) or Fermi $\eta = -1$) gas.

Ex. 11.4 *Show that the loop-density is independent of the choice of the loop parametrization.*

Replace $\mathcal{L}_a$ by $\mathcal{L}_a^{[u]}$ in the definition (11.22). Due to the invariance (H.10) the integrated loops $\mathcal{L}_i$ can be changed into $\mathcal{L}_i^{[u]}$. But the energy and the activities are invariant under the change $\mathcal{L}_i^{[u]} \to \mathcal{L}_i$.

Ex. 11.5 *Express the number density (11.28) within the loop formalism.*

One uses the invariance of the loop density (11.24) to fix the origin of the loop in $\rho_{\Lambda,\mathrm{Loop}}$ occurring in (11.28) on the l^{th} particle

$$\rho_{\Lambda,\mathrm{Loop}}(\mathbf{r} - \lambda_\alpha \boldsymbol{\mathcal{X}}(l-1), \alpha, q, \boldsymbol{\mathcal{X}}) = \rho_{\Lambda,\mathrm{Loop}}(\mathbf{r}, \alpha, q, \boldsymbol{\mathcal{X}} - \boldsymbol{\mathcal{X}}(l-1)) \qquad (84)$$

The result follows from the invariance (H.10) of $D_q(\boldsymbol{\mathcal{X}})$ under the shift $\boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}} - \boldsymbol{\mathcal{X}}(l-1)$.

Ex. 11.6 *By using (11.28), recover the standard expressions of the density in terms of the fugacity* $\mathrm{e}^{\beta\mu}$ *for ideal gases obeying either Bose-Einstein or Fermi-Dirac statistics .*

For a free gas, one deduces from (11.22) and (11.10) that $\rho(\mathcal{L}) = z(\mathcal{L}) = (2s+1)\frac{\eta^{q-1}}{q}\frac{\mathrm{e}^{\beta\mu q}}{(2\pi q\lambda^2)^{3/2}}$. Hence with $\int D_q(\boldsymbol{\mathcal{X}}) = 1$ and (11.28):

$$\rho = (2s+1)\sum_{q=1}^{\infty} q \int D_q(\boldsymbol{\mathcal{X}}) \frac{\eta^{q-1}}{q}\frac{\mathrm{e}^{\beta\mu q}}{(2\pi q\lambda^2)^{3/2}} = \frac{2s+1}{(2\pi)^3}\int \mathrm{d}\mathbf{k} \sum_{q=1}^{\infty} \eta^{q-1}\mathrm{e}^{-\beta q[\epsilon(\mathbf{k})-\mu]}$$
$$= \frac{2s+1}{(2\pi)^3}\int \mathrm{d}\mathbf{k} \frac{1}{\mathrm{e}^{\beta[\epsilon(\mathbf{k})-\mu]} - \eta} \qquad (85)$$

which is the density of ideal Bose ($\eta = 1$) or Fermi ($\eta = -1$) gases.

Ex. 11.7 *(i) Derive formula (11.31) for the exchange term. (ii) Show that, for a non-interacting electron gas, it reduces to the standard expression*

$$\rho^{\mathrm{ex}}(\mathbf{r}_\mathrm{a},\mathbf{r}_\mathrm{b}) = -2\left(\frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}_\mathrm{a}-\mathbf{r}_\mathrm{b})}\frac{1}{1+\mathrm{e}^{\beta[\epsilon(\mathbf{k})-\mu]}}\right)^2 \qquad (86)$$

which involves the square of the off-diagonal element of the one-body density matrix.

(i) The product $\hat\rho(\mathbf{r}_a)\hat\rho(\mathbf{r}_b)$ involves a sum on loops $\mathcal{L}_i, \mathcal{L}_j$. The sum on distinct loops $\sum_{i\neq j}$ leads to (11.32), whereas the diagonal part of the sum $\sum_{i=j}$ gives

$$\delta_{\alpha_a\alpha_b}\sum_i \delta_{\alpha_i\alpha_a}\left\langle \sum_{l\neq l'=1}^{q_i}\delta(\mathbf{x}_{i,l}-\mathbf{r}_a)\delta(\mathbf{x}_{i,l'}-\mathbf{r}_b)\right\rangle_{\mathrm{Loop}} =$$
$$\delta_{\alpha_a\alpha_b}\sum_{q_a=2}^{\infty}\int D(\boldsymbol{\mathcal{X}}_a)\sum_{l\neq l'=1}^{q_a}\delta[\mathbf{r}_a - \mathbf{r}_b + \lambda_a(\boldsymbol{\mathcal{X}}_a(l'-1) - \boldsymbol{\mathcal{X}}_a(l-1))]$$
$$\times\, \rho_{\Lambda,\mathrm{Loop}}(\mathbf{r}_a - \lambda_a\boldsymbol{\mathcal{X}}_a(l-1), \alpha_a, q_a, \boldsymbol{\mathcal{X}}_a) \qquad (87)$$

Due to the invariance (H.10), $\rho_{\Lambda,\mathrm{Loop}}(\mathbf{r}_a - \lambda_a \boldsymbol{\mathcal{X}}_a(l-1), \alpha_a, q_a, \boldsymbol{\mathcal{X}}_a) = \rho_{\Lambda,\mathrm{Loop}}(\mathbf{r}_a, \alpha_a, q_a, \boldsymbol{\mathcal{X}}_a - \lambda_a \boldsymbol{\mathcal{X}}_a(l-1))$ and then use (H.9) to obtain (11.31).

(ii) The solution to this exercise will be completed as soon as possible.

Ex. 11.8 *Show that the multipole part (11.17) $W(\mathcal{L},\mathcal{L}')$ of the pair loop interactions provides vanishing boundary contributions in the TL (11.35) to each Mayer graph.*

The solution to this exercise will be completed as soon as possible.

Ex. 11.9 *Show that the screening factors $\kappa^2(k,n)$ remain positive for fermions in the small-activity regime.*

The solution to this exercise will be completed as soon as possible.

Ex. 11.10 *Show that the frequency series $\sum_{n=-\infty}^{\infty} 4\pi/(k^2+\kappa^2(k,n))$ converge uniformly with respect to k in any interval $[k_0,\infty]$ with $k_0>0$.*

Since $\kappa^2(k,n)>0$, one has the majoration

$$
\begin{aligned}
&|\tilde{\phi}(\mathbf{k},\chi_a,\chi_b) - \tilde{V}(\mathbf{k},\chi_a,\chi_b)| \\
&= \left| \int_0^{q_a} \mathrm{d}s_a \int_0^{q_b} \mathrm{d}s_b \mathrm{e}^{i\mathbf{k}\cdot[\lambda_\alpha \boldsymbol{\mathcal{X}}(s_a) - \lambda_\alpha \boldsymbol{\mathcal{X}}(s_b)]} \sum_{n=-\infty}^{\infty} \frac{4\pi\kappa^2(k,n)}{k^2(k^2+\kappa^2(k,n))} \mathrm{e}^{2i\pi n(s_a-s_b)} \right| \\
&\leq q_a q_b \frac{4\pi}{k^4} \sum_{n=-\infty}^{\infty} \kappa^2(k,n), \qquad k>0
\end{aligned}
\tag{88}
$$

From (11.47)

$$
\sum_{n=-\infty}^{\infty} \kappa^2(k,n) = 4\pi\beta \int \mathrm{d}\chi e_\alpha^2 z(\chi) q \int_0^q \mathrm{e}^{i\mathbf{k}\cdot\lambda_\alpha \boldsymbol{\mathcal{X}}(s)} \tilde{\delta}(s) \leq 4\pi\beta \int \mathrm{d}\chi e_\alpha^2 |z(\chi)| q^2
\tag{89}
$$

Since the interaction of the particles of the same species is repulsive, one has $U_{\mathrm{self}}>0$ and $|z(\chi)| \leq \frac{2s_\alpha+1}{q} \frac{\mathrm{e}^{\beta\mu q}}{(2\pi q\lambda_\alpha^2)^{3/2}}$ (see (11.10), the above series converge for $\mathrm{e}^{\beta\mu}<1$ uniformly with respect to k. This implies that $\tilde{\phi}(\mathbf{k},\chi_a,\chi_b)$ is a continuous function of $\mathbf{k}$, $k>0$.

Ex. 11.11 *Show that the summation of chains with V_{elec} (11.16) yields the zero-frequency term of the effective potential.*

The result is immediate if one recalls that $V_{\mathrm{elec}}(\mathcal{L}_i,\mathcal{L}_j)$ (11.15) is the term $n=0$ in the frequency decomposition of the of the loop potential. It suffices to go through the derivation of the effective potential in the Complement 11.1 keeping only the terms $n=0$.

Ex. 11.12 *Show that quantum effects can be neglected at intermediate distance $r\sim\ell_S$ in the low-activity regime.*

The solution to this exercise will be completed as soon as possible.

Ex. 11.13 *Calculate the RPA screening factor and derive the small-k behavior (11.86).*

The solution to this exercise will be completed as soon as possible.

Ex. 11.14 *Determine the RPA screening length ℓ_{RPA} which controls the exponential decay of the zero-frequency component $\tilde{\psi}_{\mathrm{root}}(r,0)$, in the two limits: (i) $z_\alpha \to 0$ at fixed T (ii) $T \to \infty$ at fixed z_α. Show that, in both limits, the RPA and Thomas-Fermi screening lengths become equivalent to the Debye classical screening length, $\ell_{\mathrm{RPA}} \sim \ell_{\mathrm{TF}} \sim \ell_{\mathrm{D}}$.*

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 12

Ex.12.1 *Determine the positions of the poles in the complex k-plane of $\tilde{\phi}^{\mathrm{reg}}$, (i) for the static part, (ii) for the regular dynamic part.*

The solution to this exercise will be completed as soon as possible.

Ex.12.2 *Calculate the coefficient $c^{\mathrm{MB}}_{z,n}$ (12.18).*

The first equality (12.18) follows immediately from (11.65) where $q=1$, $\mathcal{L}=\boldsymbol{\xi}$ reduces to a filament and $z(\boldsymbol{\xi})$ is identified to the density ρ_α in the Maxwell-Boltzmann regime. One deduces from (G.7) that $\int D(\boldsymbol{\xi})\boldsymbol{\xi}^2(s) = 3(s-s^2)$ and by integration by parts

$$\int_0^1 \mathrm{d}s\ \mathrm{e}^{2i\pi ns}(s-s^2) = -\frac{1}{2\pi^2 n^2} \tag{90}$$

Hence the result (12.18).

Ex.12.3 *Write down the analytical expressions of the graphs in Figure 12.3 corresponding to the Debye dressed family.*

The solution to this exercise will be completed as soon as possible.

Ex.12.4 *(i) Show that the Fourier transform of $|\mathbf{r}|^{-6}$ equals $(\pi/12)|\mathbf{k}|^3$. (ii) Derive the expression (12.39) by identifying the singular term in the Fourier transform of the convolution (12.34).*

(i) In spherical coordinates with $u=\cos\theta$

$$\begin{aligned}\int \mathrm{d}\mathbf{r}\frac{\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}}}{r^6} = 2\pi\int_0^\infty \mathrm{d}r\frac{1}{r^4}\int_{-1}^1 \mathrm{d}u\mathrm{e}^{-ikru} &= \frac{2\pi}{-ik}\int_0^\infty \mathrm{d}r\frac{\mathrm{e}^{-ikr}-\mathrm{e}^{ikr}}{r^5}\\ &= \frac{2\pi}{-ik}\int_{-\infty}^\infty \mathrm{d}r\frac{\mathrm{e}^{-ikr}}{r^5}\end{aligned} \tag{91}$$

The integrals under consideration are singular and should be rigorously treated in the framework of the theory of distributions. Nevertheless they can be quickly calculated starting from the following regularized integral, setting $\mathrm{e}^{-ikr} = \cos kr - i \sin kr$ ($\cos kr$ does not contribute to the integral by parity $r \to -r$)

$$\int_{-\infty}^{\infty} \mathrm{d}r \frac{\mathrm{e}^{-ikr}}{r} = \lim_{R\to\infty,\epsilon\to 0} -i \left[\int_{-R}^{-\epsilon} \frac{\sin kr}{r} + \int_{\epsilon}^{R} \frac{\sin kr}{r} \right]$$
$$= -2i \lim_{R\to\infty,\epsilon\to 0} \int_{\epsilon}^{R} \frac{\sin kr}{r} = -i\pi \mathrm{sign}(k), \quad \mathrm{sign}(k) = 1, k > 0, = -1, k < 0 \tag{92}$$

Then by integration by parts

$$\int_{-\infty}^{\infty} \mathrm{d}r \frac{\mathrm{e}^{-ikr}}{r^5} = \frac{1}{24} \int_{-\infty}^{\infty} \mathrm{d}r \frac{\mathrm{d}^4}{\mathrm{d}r^4} \left(\frac{1}{r} \right) \mathrm{e}^{-ikr} = \frac{k^4}{24} \int_{-\infty}^{\infty} \mathrm{d}r \frac{\mathrm{e}^{-ikr}}{r} = -i\pi \frac{k^4}{24} \mathrm{sign}(k), \tag{93}$$

and we get with (91)

$$\int \mathrm{d}\mathbf{r} \frac{\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}}}{r^6} = \frac{\pi^2}{12} |\mathbf{k}|^3 \tag{94}$$

(ii) By the convolution theorem, the Fourier transform of the convolution (12.34) is

$$\int \mathrm{d}(\mathbf{r}_a - \mathbf{r}_b) \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}_a - \mathbf{r}_b)} \int \mathrm{d}\mathbf{r} f(\mathbf{r}_a - \mathbf{r}) g(\mathbf{r} - \mathbf{r}_b) = \tilde{f}(\mathbf{k}) \tilde{g}(\mathbf{k}) \tag{95}$$

The small $\mathbf{k}$ expansion of $\tilde{f}(\mathbf{k}) = \tilde{b}_{\mathrm{AM}}(\mathbf{k})$ (omitting the filament shape variables) is of the form (see (14.18))

$$\tilde{f}(\mathbf{k}) = \mathcal{P}(\mathbf{k}) + \mathcal{B}(\hat{k}) \frac{\pi^2}{12} |\mathbf{k}|^3 + \cdots \tag{96}$$

where $\mathcal{P}(\mathbf{k})$ is a quadratic form in $\mathbf{k}$ (see (14.19)) and the singular $|\mathbf{k}|^3$ term arises from the from the r^{-6} decay of $b_{\mathrm{AM}}(\mathbf{r})$ (see (12.38) or (14.13)). From the analysis of Section 12.2, one knows that $g(\mathbf{r})$ decays faster than any inverse power of $\mathbf{r}$, therefore the $\mathbf{k}$ expansion of $\tilde{g}(\mathbf{k})$ has no singular term and by (12.37), and $\tilde{g}(\mathbf{k} = 0) = -\beta e_a e_b \frac{4\pi}{\kappa_{\mathrm{D}}^2}$. Hence the most singular term in the $|\mathbf{k}|$ expansion of $\tilde{f}(\mathbf{k})\tilde{g}(\mathbf{k})$ is

$$-\beta e_a e_b \frac{4\pi}{\kappa_{\mathrm{D}}^2} \mathcal{B}(\hat{k}) \frac{\pi^2}{12} |\mathbf{k}|^3 \tag{97}$$

Taking the inverse Fourier transform of (97), introducing the filament shape variables densities and working out the coefficient $\mathcal{B}$ at low density as in the main text, one recovers (12.39) and (12.40) .

Ex.12.5 *Calculate the amplitude of the tails of the particle-charge and charge-charge correlations.*

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 13

Ex.13.1 *Show that the integral of the dynamical components $\phi^{(n\neq 0)}$ vanishes.*

The solution to this exercise will be completed as soon as possible.

Ex. 13.2 *Establish formula (13.47).*

The solution to this exercise will be completed as soon as possible.

Ex. 13.3 *Show that the $z^{3/2}$ terms in expansion (13.50) do satisfy local charge neutrality.*

The solution to this exercise will be completed as soon as possible.

Ex. 13.4 *Express the activities z_α in terms of the densities ρ_α.*

The solution to this exercise will be completed as soon as possible.

Ex. 13.5 *Show that, within the scaling limit $\zeta \to 0$, the only non-vanishing contribution to the charge density $e_2\rho_2$ in the whole series of TCP prototype graphs for $\rho(\mathcal{L}_{\mathrm{a}}^{(2)})$ arises from the simplest diagram made of the single root loop $\mathcal{L}_{\mathrm{a}}^{(2)}$, and it reduces to $-e\rho_{\mathrm{B}}$.*

The solution to this exercise will be completed as soon as possible.

Ex. 13.6 *Show that, in the loop gas representation of the TCP, the scaling limit $\zeta \to 0$ of each term in the grand canonical sum, reduces to its OCP counterpart. Hint: integrate first over the degrees of freedom of species $\alpha = 2$ within a cumulant expansion.*

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 14

Ex.14.1 *Calculate the amplitude of order $\rho^{5/2}$ in (14.53) for the hydrogen plasma.*

The solution to this exercise will be completed as soon as possible.

Ex. 14.2 *Calculate the effect of quantum statistics on the Abe-Meeron amplitude (14.17) to order ρ^3 in the high temperature regime.*

The solution to this exercise will be completed as soon as possible.

Ex. 14.3 *Calculate the covariance of the Brownian bridge process in presence of a uniform magnetic field.*

See appendices in [2]). According to (14.118), the magnetic phase factor for $\mathbf{B}$ in the z-direction reads

$$iw\int_0^1 [\xi_x(s)\mathrm{d}\xi_y(s) - \xi_y(s)\mathrm{d}\xi_x(s)] \qquad w = \frac{e\lambda^2}{2\hbar c}B = \frac{\beta\hbar e B}{2mc} \tag{98}$$

We look for the two-dimensional Gaussian measure

$$D(\boldsymbol{\xi}_x)\left[D(\boldsymbol{\xi}_y)\exp\left(iw\int_0^1[\xi_x(s)\mathrm{d}\xi_y(s)-\xi_y(s)\mathrm{d}\xi_x(s)]\right)\right] \tag{99}$$

and we aim to cast it into a one-dimensional measure of the form

$$D(\boldsymbol{\xi}_x)\exp\left(-\frac{1}{2}\int_0^1\mathrm{d}s_1\int_0^1\mathrm{d}s_2\xi_x(s_1)A(s_1,s_2)\xi_x(s_2)\right) \tag{100}$$

by explicitly performing the $D(\boldsymbol{\xi}_y)$ integration in (99). We remember that the covariance of the Gaussian measure $D(\xi)$ is the inverse of the operator$-\frac{\mathrm{d}^2}{\mathrm{d}s^2}$(see (G.9), (G.10)) so that (100) can formally be written as the Brownien integral (up to a normalisation factor)

$$\mathrm{d}\xi\exp\left(-\frac{1}{2}\int_0^1\mathrm{d}s_1\int_0^1\mathrm{d}s_2\xi_x(s_1)\left[-\delta(s_1-s_2)\frac{\mathrm{d}^2}{\mathrm{d}s_1^2}+A(s_1,s_2)\right]\xi_x(s_2)\right) \tag{101}$$

According to the Gaussian rules the covariance in the presence of the field is the inverse of the operator defined by the square bracket in (101)

$$-\frac{\mathrm{d}^2}{\mathrm{d}s_1^2}[\mathrm{cov}]_{\mathrm{B}}(s_1,s_2)+\int_0^1\mathrm{d}sA(s_1,s)[\mathrm{cov}]_{\mathrm{B}}(s,s_2)=\delta(s_1-s_2) \tag{102}$$

It remains to determine $A(s_1,s_2)$.The expression $\xi_x(s)\mathrm{d}\xi_y(s)-\xi_y(s)\mathrm{d}\xi_x(s)$ is linear function of ξ_y ($\mathrm{d}\xi_y(s)=\xi_y(s+\epsilon)-\xi_y(s)$. It can be written in the form $\int_0^1 f(s)\xi_y(s)$ where $f(s)$ depends on $\xi_x(s)$ and $\mathrm{d}\xi_x(s)$. From the Gaussian rule (C.10), one finds that the square bracket in (99) is equal to

$$[\cdots]=\exp\left(-\frac{1}{2}\int_0^1\int_0^1 f(s_1)[\mathrm{cov}](\mathrm{s}_1,\mathrm{s}_2)\mathrm{f}(\mathrm{s}_2)\right)=\exp\left(-\frac{1}{2}w^2\int D(\xi_y)|F(\xi_y)|^2\right) \tag{103}$$

where $[\mathrm{cov}](\mathrm{s}_1,\mathrm{s}_2)$ is the covariance (G.7) of the brownian bridge and

$$F(\xi_y)=\int_0^1[\xi_x(s)\mathrm{d}\xi_y(s)-\xi_y(s)\mathrm{d}\xi_x(s)] \tag{104}$$

Expanding the exponent in (103) yields 4 terms

$$\begin{aligned}&\int_0^1\int_0^2\int D(\xi_y)[\xi_x(s_1)\xi_x(s_2)\mathrm{d}\xi_y(s_1)\mathrm{d}\xi_y(s_2)-\xi_x(s_1)\mathrm{d}\xi_x(s_2)\mathrm{d}\xi_y(s_1)\xi_y(s_2)\\&-\xi_x(s_1)\mathrm{d}\xi_x(s_2)\xi_y(s_1)\mathrm{d}\xi_y(s_2)+\mathrm{d}\xi_x(s_1)\mathrm{d}\xi_x(s_2)\xi_y(s_1)\xi_y(s_2)]\end{aligned} \tag{105}$$

One calculates

$$\begin{aligned}&\int D(\xi_y)\mathrm{d}\xi_y(s_1)\mathrm{d}\xi_y(s_2)=\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\int D(\xi_y))\xi_y(s_1))\xi_y\mathrm{d}s_1\mathrm{d}s_2\\&=\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}[\mathrm{cov}](\mathrm{s}_1,\mathrm{s}_2)\mathrm{ds}_1\mathrm{ds}_2=(\delta(\mathrm{s}_1-\mathrm{s}_2)-1)\mathrm{ds}_1\mathrm{ds}_2\end{aligned} \tag{106}$$

so that the first term of (105) is

$$\int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \xi_x(s_1)(\delta(s_1 - s_2) - 1)\xi_x(s_2) \tag{107}$$

In the same way

$$\int D(\xi_y)\mathrm{d}\xi_y(s_1)\xi_y(s_2) = \frac{\partial}{\partial s_1}[\mathrm{cov}](\mathrm{s}_1, \mathrm{s}_2)\mathrm{ds}_1 = (\theta(\mathrm{s}_2 - \mathrm{s}_1) - \mathrm{s}_2)\mathrm{ds}_1 \tag{108}$$

so that the second term of (105) is identical to (107)

$$\begin{aligned} &- \int_0^1 \mathrm{d}s_1 \int_0^1 \xi_x(s_1) \int \mathrm{d}\xi_x(s_2)(\theta(s_2 - s_1) - s_2) \\ &= \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \xi_x(s_1)\xi_x(s_2) \frac{\partial}{\partial s_2}(\theta(s_2 - s_1) - s_2) \\ &= \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \xi_x(s_1)(\delta(s_1 - s_2) - 1)\xi_x(s_2) \end{aligned} \tag{109}$$

where we have applied Ito's lemma, $\int_0^1 \mathrm{d}\xi(s)f(s) = -\int_0^1 \xi(s)\frac{\mathrm{d}}{\mathrm{d}s}f(s)$, $\xi(0) = \xi(1) = 0$. One checks by similar calculations that all terms in (105) give the same result, hence (99) is equal to

$$\int D(\xi_x) \exp\left(-2w^2 \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \xi_x(s_1)(\delta(s_1 - s_2) - 1)\xi_x(s_2)\right) \tag{110}$$

Hence, in (101), $A(s_1, s_2) = 4w^2(\delta(s_1 - s_2) - 1)$ and (102) reads

$$-\frac{\mathrm{d}^2}{\mathrm{d}s_1^2}[\mathrm{cov}]_{\mathrm{B}}(s_1, s_2) + 2w^2\left([\mathrm{cov}]_{\mathrm{B}}(s_1, s_2) - \int_0^1 \mathrm{d}s[\mathrm{cov}]_{\mathrm{B}}(s,s_2)\right) = \delta(s_1 - s_2) \tag{111}$$

The solution of this integral equation with conditions $[\mathrm{cov}]_{\mathrm{B}}(0, s_2) = [\mathrm{cov}]_{\mathrm{B}}(s_1, 0) = 0$ is

$$\begin{aligned} [\mathrm{cov}]_{\mathrm{B}}(s_1, s_2) = &\frac{\sinh(2ws_<)\sinh(2w(1 - s_>))}{2w\sinh(2w)} \\ &+ \frac{\left(1 - \frac{\cosh(2w(s_1 - 1/2))}{\cosh(w)}\right)\left(1 - \frac{\cosh(2w(s_2 - 1/2))}{\cosh(w)}\right)}{4w\tanh(w)} \\ &s_< = \min(s_1, s_2) \qquad s_> = \max(s_1, s_2) \end{aligned} \tag{112}$$

which reduces to Equation (14.129), using a number of hyperbolic identities

$$[\mathrm{cov}]_{\mathrm{B}}(s, s) = \frac{\sinh ws \sinh w(1 - s)}{w \sinh w} \tag{113}$$

when $s_1 = s_2 = s$.

NOTE. The solution of Equation (111) given in [2] ((i.e. Equation (47) of [2]) is not correct for $s_1 \neq s_2$ (a second term as in the r.h.s. of (112) is missing.).

Nevertheless Equation (47) of [2] agrees with (113) when $s_1 = s_2 = s$ so that the amplitude (14.131) is right. All these calculations can be checked wit IA. One finds in particular that

$$A_{\mathrm{B}}^{(0)}(\alpha_a, \alpha_b) = -\beta \rho_{\alpha_{\mathrm{a}}} \rho_{\alpha_{\mathrm{b}}} e_{\alpha_{\mathrm{a}}} e_{\alpha_{\mathrm{b}}} \lambda_{\alpha_{\mathrm{a}}}^2 \lambda_{\alpha_{\mathrm{b}}}^2 \frac{w_a^2 w_b^2}{12600} + \mathrm{O}(w^6) \sim \hbar^8 B^4, \quad B \to 0 \tag{114}$$

$$\begin{aligned} A_{\mathrm{B}}^{(0)}(\alpha_a, \alpha_b) &= -\frac{1}{30} \beta \rho_{\alpha_{\mathrm{a}}} \rho_{\alpha_{\mathrm{b}}} e_{\alpha_{\mathrm{a}}} e_{\alpha_{\mathrm{b}}} \lambda_{\alpha_{\mathrm{a}}}^2 \lambda_{\alpha_{\mathrm{b}}}^2 + \frac{3}{2 w_a w_b} + \mathrm{O}\left(\frac{1}{w^3}\right) \\ &= -\frac{1}{30} \beta \rho_{\alpha_{\mathrm{a}}} \rho_{\alpha_{\mathrm{b}}} e_{\alpha_{\mathrm{a}}} e_{\alpha_{\mathrm{b}}} \lambda_{\alpha_{\mathrm{a}}}^2 \lambda_{\alpha_{\mathrm{b}}}^2 + \mathrm{O}\left(\frac{1}{\hbar^2 B^2}\right), \quad B \to \infty \end{aligned} \tag{115}$$

Exercices of Chapter 15

Ex. 15.1 *Derive the approximate expression (15.6).*

The partition function of the hydrogen atom (with a proton at rest) in volume Λ is

$$Z_\Lambda = \sum_{E_{\mathrm{bound}}} \mathrm{e}^{-\beta E_{\mathrm{bound}}} + \sum_{E_{\mathrm{ion}}} \mathrm{e}^{-\beta E_{\mathrm{ion}}} > \sum_{E_{\mathrm{ion}}} \mathrm{e}^{-\beta E_{\mathrm{ion}}} \tag{116}$$

The sums run on the energies of bound states $E_{\mathrm{bound}} < 0$ and ionized states $E_{\mathrm{ion}} > 0$ of the electron. For the sake of simplicity and to obtain an order of magnitude, assimilate the ionized states to free electronic states in a cube of side L with energies $\frac{\hbar^2}{2m}|\mathbf{k_n}|^2$, $\mathbf{k_n} = \frac{\pi n_1}{L}, \dots, \frac{\pi n_\nu}{L}\}$, $\mathbf{n} = \{n_1, \dots, n_\nu = 1, 2, \dots\}$ corresponding to Dirichlet conditions [4]. Then for L large

$$\sum_{E_{\mathrm{ion}}} \mathrm{e}^{-\beta E_{\mathrm{ion}}} = \sum_{n_1, n_2, n_3 > 0} \mathrm{e}^{-\beta \frac{\hbar^2}{2m_e}|\mathbf{k}|^2} \sim \left(\frac{L}{2\pi}\right)^3 \int \mathrm{d}\mathbf{k}\, \mathrm{e}^{-\beta \frac{\hbar^2}{2m}|\mathbf{k}|^2} = \left(\frac{L}{\sqrt{2\pi}\lambda}\right)^3 \tag{117}$$

Ex. 15.2 *Derive the atomic density (15.9).*

The grand partition function (4.37) corresponding to the Hamiltonian (15.8) reduces to

$$\Xi_\Lambda = \exp\left[\mathrm{e}^{\beta \mu_H} \mathrm{Tr}_\Lambda \mathrm{e}^{-\beta H^{\mathrm{atom}}}\right] \tag{118}$$

Evaluating the trace in a box Λ of side L with periodic boundary conditions gives

$$\mathrm{Tr}_\Lambda \mathrm{e}^{-\beta H^{\mathrm{atom}}} \sim 4 \left(\frac{m_e + m_p}{2\pi \beta \hbar^2}\right)^{3/2} \mathrm{e}^{-\beta E_H} L^3, \quad L \to \infty \tag{119}$$

[4] Extended Coulombic states are plane waves with a logarithmic correction [5], which will not change the conclusion (15.6).

The factor 4 accounts for the 4 degenerate spin states of the electron-proton pair. Thus the pressure is equal to

$$\beta P^{\mathrm{id}} = \lim_{L\to\infty} \frac{1}{L^3} \ln \Xi_\Lambda = \rho_H^{\mathrm{id}} \tag{120}$$

where $\rho_H^{\mathrm{id}} = \frac{\partial}{\partial \mu_H} P^{\mathrm{id}}$ is given by (15.9).

Ex. 15.3 *Retrieve (15.16) through a thermodynamic derivation in the canonical ensemble.*

The solution to this exercise will be completed as soon as possible.

Ex. 15.4 *Derive (15.23), (15.24) and (15.25).*

(15.23) follows from the definitions after a little algebra. From $\rho = \rho^* \left(2\gamma + \gamma^2\right)$ one finds $\gamma = \sqrt{1 + \frac{\rho}{\rho^*}} - 1$ and hence (15.25) by substitution in (15.23).

Ex. 15.5 *Derive the lower bound (15.41) for $E_{N_\mathrm{e} N_\mathrm{p}}$.*

If $(N_e, N_p) = (0,1)$ or $(1,0)$, obviously $E_{01} \geq 0$ and $E_{10} \geq 0$. Moreover if (15.37) holds, set $\mu = E_H + \epsilon$ for some $\epsilon > 0$. Then

$$\begin{aligned} E_{N_e N_p} &> (E_H + \epsilon)(N_e + N_p - 1) - \epsilon \\ &= \left(E_H + \frac{\epsilon}{2}\right)(N_e + N_p - 1) + \frac{\epsilon}{2}(N_e + N_p - 3) \\ &\geq \left(E_H + \frac{\epsilon}{2}\right)(N_e + N_p - 1) \quad \text{when } N_e + N_p \geq 3. \end{aligned} \tag{121}$$

This provides the inequality (15.41) with $k = |E_H| - \frac{\epsilon}{2} < |E_H|$.

Conversely, if (15.41) holds, one can set $k = |E_H| - \epsilon = -E_H - \epsilon$, $\epsilon > 0$, and if $(N_e,N_p) \neq (0{,}0)$, (11)

$$E_{N_e N_p} - (E_H + \epsilon)(N_e + N_p) \geq -E_H - \epsilon \tag{122}$$

Setting $\mu = E_H + \frac{\epsilon}{2}$ gives

$$E_{N_e N_p} - \mu(N_e + N_p) \geq E_H - 2\mu + \frac{\epsilon}{2}(N_e + N_p) > E_H - 2\mu \tag{123}$$

Thus one can find $\mu > E_H$ such that the inequalities (15.37) are verified.

Ex. 15.6 *Show that the excited states of the hydrogen atom provide contributions to $\mathrm{e}^{2\beta\mu} Z(\mathcal{D}, 1, 1, \beta)$ which are exponentially smaller than (15.51) in the atomic limit.*

Denote the Hamiltonian (15.48) of the hydrogen atom projected on the subspace of excited states by $\bar{H}_{11} = K_{11}^{\mathrm{cm}} + \bar{H}_{11}^{\mathrm{rel}}$ (this projection does not affect the center of mass coordinates). Using the same scaling transformation as in (15.56), one has

$$\begin{aligned}&\text{infspectrum}[(1-\eta)\mathrm{K}_{11}^{\mathrm{cm}}+\bar{H}_{11}^{\mathrm{rel}}]=(1-\eta)^{-1}\text{infspectrum }\bar{H}_{11}\\&=(1-\eta)^{-1}(E_H+\Delta)=E_H+\Delta'\end{aligned}\tag{124}$$

where Δ is the energy gap between the ground state and the first excited state and Δ' is positive provided that η is sufficiently small. Thus

$$\bar{H}_{11}=\eta K_{11}^{\mathrm{cm}}+(1-\eta)K_{11}^{\mathrm{cm}}+\bar{H}_{11}^{\mathrm{rel}}\geq\eta K_{11}^{\mathrm{cm}}+E_H+\Delta'\tag{125}$$

The partition function restricted to the excited states is bounded as in (15.51) by

$$\mathrm{e}^{2\beta\mu}Z_{\mathrm{exc}}(B,1,1,\beta)\leq 4\left(\frac{m_e+m_p}{2\pi\eta\beta\hbar^2}\right)^{3/2}\mathrm{e}^{-\beta(E_H-2\mu)}|B|\mathrm{e}^{-\beta\Delta'}\leq\rho_H^{\mathrm{id}}|B|\mathrm{e}^{-\beta\Delta''}\tag{126}$$

where η has been replaced by 1 at the cost of some smaller Δ'', $0<\Delta''<\Delta'$.

Ex. 15.7 *Derive the lower bound (15.69) for $\Xi(\Lambda,\boldsymbol{\mu},\beta)$ when subdomains $\mathcal{D}_j$ are balls.*

Take the case of two balls B_1 and B_2 in Λ. Forming the grand canonical sum, the subdomain inequality (5.56) yields

$$\begin{aligned}\Xi(\beta,\Lambda,\boldsymbol{\mu})&\geq\sum_{\mathbf{N}_1}\sum_{\mathbf{N}_2}\mathrm{e}^{\beta\boldsymbol{\mu}\cdot\mathbf{N}_1}Z(\beta,B_1,\mathbf{N}_1)\mathrm{e}^{\beta\boldsymbol{\mu}\cdot\mathbf{N}_2}Z(\beta,B_2,\mathbf{N}_2)\mathrm{e}^{-\beta\langle W_{\mathbf{N}_1\mathbf{N}_2}\rangle_{12}}\\&\geq 1+\mathrm{e}^{2\mu}Z(\beta,B_1,1,1)+\mathrm{e}^{2\mu}Z(\beta,B_2,1,1)\end{aligned}\tag{127}$$

where one has kept only the hydrogen partition functions. Because of spherical symmetry and neutrality these two balls do not interact as a consequence of Newton's theorem. In view of (15.51) we obtain

$$\Xi(\beta,\Lambda,\boldsymbol{\mu})\geq 1+\rho_H^{\mathrm{id}}(|B_1|+|B_2|)\left(1+\mathrm{O}\left(\mathrm{e}^{-b\beta/3}\right)\right)\tag{128}$$

Replacing balls by more general domains $\mathcal{D}_j$, e.g. cubes, requires some additional work, but (128) generalizes to

$$\Xi(\beta,\Lambda,\boldsymbol{\mu})\geq 1+\rho_H^{\mathrm{id}}\sum_j|\mathcal{D}_j|\left(1+\mathrm{O}\left(\mathrm{e}^{-b\beta/3}\right)\right)=1+\rho_H^{\mathrm{id}}|\Lambda|\left(1+\mathrm{O}(\mathrm{e}^{-c\beta})\right)\tag{129}$$

This proves the reversed inequality (15.68).

Ex. 15.8 *Show that the common leading behavior (15.99) of both ρ_{e} and ρ_{p} yields the Saha expression of the pressure.*

The solution to this exercise will be completed as soon as possible.

Ex. 15.9 *Show that the shift correction (15.111) provides contributions to (15.105) exponentially smaller than* $\rho_{\mathrm{H}}^{\mathrm{id}}$.

The solution to this exercise will be completed as soon as possible.

Ex. 15.10 *Show that the application of the Feynman-Kac formula backwards to the functional integral of (15.117) provides the spatial matrix elements (15.120).*

The solution to this exercise will be completed as soon as possible.

Ex. 15.11 *Enumerate the diagrams analog to Figure 15.4 which embed the contribution of* H_2^+ *to* ρ_{e} *and show that their leading contribution in the Saha regime reduces to* $\rho_{\mathrm{H}_2^+}^{\mathrm{id}}$.

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 16

Ex. 16.1 *Show that the Debye dressing of diagrams (Figure 16.1) provides exponentially smaller contributions in the Saha regime.*

The solution to this exercise will be completed as soon as possible.

Ex. 16.2 *Derive the correction (16.62) to the leading behavior of* $[\epsilon(s) - 1]$ *in the Saha regime.*

The solution to this exercise will be completed as soon as possible.

Ex. 16.3 *Extract from (16.63) a contribution to the amplitude of the effective atom-atom potential with a structure similar to (16.29), involving now the product of the fourth moment of the fluctuating internal dipole of* $\mathcal{C}_{\mathrm{H}}$ *times the second moments of the dipoles relative to* $\mathcal{C}_{\mathrm{H}}^a$ *and* $\mathcal{C}_{\mathrm{H}}^b$. *Show that this contribution is exponentially smaller than (16.29) by a factor of order* $\mathrm{e}^{\beta E_{\mathrm{H}}}$.

The solution to this exercise will be completed as soon as possible.

Ex. 16.4 *Derive the force (16.68) between two slabs due to van der Waals interactions.*

Two slabs A, B of thickness a and b, perpendicular to the x-axis, are separated by the distance d. Their surfaces L^2 are located at $x = 0$ and $x = d$. Let $\mathbf{y}, \mathbf{y}'$ be the two dimensional coordinates along the slabs. Because of translation invariance in the $\mathbf{y}$ direction, the average force f between the slabs per unit surface will be along the x axis. Therefore it is sufficient to calculate the x component of the van der Waals force between two atoms at $\mathbf{x} = (x, \mathbf{y})$ in A and $\mathbf{x}' = (x', \mathbf{y}')$ in B

$$
\begin{aligned}
F &= -\mathcal{A}_{\mathrm{vdW}} \frac{\partial}{\partial(x - x')} \left[\frac{1}{\sqrt{(x - x')^2 + (\mathbf{y} - \mathbf{y}')^2}} \right]^6 \\
&= 6\mathcal{A}_{\mathrm{vdW}} \left[\frac{x - x'}{(x - x')^2 + |\mathbf{y} - \mathbf{y}'|^2} \right]^4
\end{aligned} \tag{130}
$$

Considering that the densities of the slabs $\rho_{\rm A}, \rho_{\rm B}$ are uniform, the force between the slabs per unit surface is

$$f = \frac{\rho_{\rm A}\rho_{\rm B}}{L^2} 6\mathcal{A}_{\rm vdW} \int_{-a}^{0} {\rm d}x \int_{\rm d}^{\rm d+b} {\rm d}x' \int_{\rm L^2} {\rm d}\mathbf{y} \int_{\rm L^2} {\rm d}\mathbf{y}' \left[\frac{x-x'}{(x-x')^2 + |\mathbf{y}-\mathbf{y}'|^2}\right]^4 \tag{131}$$

In the limit of infinity extended slabs $L \to \infty$, the $\mathbf{y}$,$\mathbf{y}'$ integrals tend to

$$\begin{aligned} &\lim_{\rm L\to\infty} \frac{1}{\rm L^2} \int_{\rm L^2} {\rm d}\mathbf{y} \int_{\rm L^2} {\rm d}\mathbf{y}' \left[\frac{1}{(x-x')^2 + |\mathbf{y}-\mathbf{y}'|^2}\right]^4 \\ &\qquad = \int {\rm d}\mathbf{y} \left[\frac{1}{(x-x')^2 + |\mathbf{y}|^2}\right]^4 = \frac{\pi}{3(x-x')^6} \end{aligned} \tag{132}$$

so that for thick slabs $a, b >> d$

$$f = 2\pi\rho_{\rm A}\rho_{\rm B}\mathcal{A}_{\rm vdW} \int_{-\infty}^{0} {\rm d}x \int_{d}^{\infty} {\rm d}x' \frac{1}{(x-x')^5} = -\pi\rho_{\rm A}\rho_{\rm B}\frac{\mathcal{A}_{\rm vdW}}{6d^3} \tag{133}$$

Exercices of Chapter 17

Ex. 17.1 *Derive the second line in (17.14).*

The solution to this exercise will be completed as soon as possible.

Ex. 17.2 *Express, in terms of Ursell functions, the short-range parts of the direct correlations for a non-symmetric two-component system including a background.*

The solution to this exercise will be completed as soon as possible.

Ex. 17.3 *Recover the terms of order $\epsilon^2 \ln \epsilon$ and ϵ^2 in the small-ϵ expansion (17.43) of the EOS by starting from the compressibility sum rule.*

The solution to this exercise will be completed as soon as possible.

Ex. 17.4 *Show that the lowest order contribution of chain diagrams $\mathcal{G}_{\rm U}^{\rm P}$ increases with N.*

The solution to this exercise will be completed as soon as possible.

Ex. 17.5 *Calculate the terms of order $\epsilon^2 \ln \epsilon$ and ϵ^2 in the small- ϵ expansion of the integral (17.46). Show that the remainder is an infinite series of terms of order either $\epsilon^n \ln \epsilon$ or ϵ^n and $n \geq 3$.*

The solution to this exercise will be completed as soon as possible.

Ex. 17.6 *Calculate the excess internal energy of the OCP within the ion-sphere model.*

The solution to this exercise will be completed as soon as possible.

Ex. 17.7 *Check that the low-density form, up to order $\rho^{5/2}$, of the β-expansion (17.70) of the pressure, does reduce to the high-temperature form, up to order $\beta^{5/2}$, of the small-ρ expansion (17.61) of the EOS.*

The solution to this exercise will be completed as soon as possible.

Ex. 17.8 *Using the reorganization of the Abe-Meeron diagrammatic series generated by the high-temperature decomposition (17.71), calculate successively the lowest-order corrections of order $\beta^{3/2}$ to the HS compressibility and pressure.*

The solution to this exercise will be completed as soon as possible.

Ex. 17.9 *Express the 2D Debye potential in terms of a Hankel function of zeroth order. Check that the corresponding contribution of the Debye diagram to $u^*_{\rm exc}(\Gamma)$ reduces to the first term in the expansion (17.83).*

The solution to this exercise will be completed as soon as possible.

Ex. 17.10 *Derive expression (17.88) for $f^*_{\rm exc}(2)$ by controlling the limit of (17.86) when $N \to \infty$.*

The solution to this exercise will be completed as soon as possible.

Ex. 17.11 *Derive expressions (17.89) and (17.90).*

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 18

Ex. 18.1 *Derive relation (18.20) for $\kappa^2_{\rm eff}$.*

The solution to this exercise will be completed as soon as possible.

Ex. 18.2 *Establish the convolution property (18.23).*

The solution to this exercise will be completed as soon as possible.

Ex. 18.3 *Show that series (13.12) for $\rho(\mathcal{L}_{\rm a})$ are recovered by using (18.28) in (18.29). Hint: the screened potential ϕ is a functional of $z(\mathcal{L})$.*

The solution to this exercise will be completed as soon as possible.

Ex. 18.4 *Show that the ring pressure (18.27) provides the Debye correction of order $\rho^{3/2}$ to the ideal pressure in (18.1).*

The solution to this exercise will be completed as soon as possible.

Ex. 18.5 *Using the WK correction (10.60) to the classical two-body distribution function, show that the $\hbar^2$-correction to the classical OCP pressure is given by (18.33).*

The solution to this exercise will be completed as soon as possible.

Ex. 18.6 *Derive formula (18.81) for a single particle in an external, static potential.*

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 19

Ex. 19.1 *Starting from the RG flow equations (19.13) and (19.14), derive the behavior of ℓ_{S} in the conducting phase near the KT line.*

The solution to this exercise will be completed as soon as possible.

Ex. 19.2 *Calculate the amplitude $A^{(4)}_{++}$ of the $1/r^4$ decay (19.70).*

The solution to this exercise will be completed as soon as possible.

Ex. 19.3 *Starting from the RG flow equations (19.13) and (19.14), derive (i) the large r behavior of $S(r)$, and (ii) the behavior of ℓ_{S} in the conducting phase near the KT line.*

The solution to this exercise will be completed as soon as possible.

Exercices of Chapter 20

Ex. 20.1 *Determine the critical density and critical temperature within the crude vdW-like approach.*

The solution to this exercise will be completed as soon as possible.

Ex. 20.2 *Derive the mass-action law (20.2) from suitable approximations in the activity-expansion of the density.*

The solution to this exercise will be completed as soon as possible.

Ex. 20.3 *Show that the contribution $\mathrm{O}(1/r^{4+\eta})$ of the three-body term in the r.h.s. of (20.34) has a vanishing amplitude.*

The solution to this exercise will be completed as soon as possible.

Exercices of Appendix B

Ex. B.1 *(i) Derive the expression (B.9) of* $\rho(1|2|3)$ *from the general formula (B.10). Does the formula* $\langle 1|2|3\rangle = \langle (1-\langle 1\rangle)(2-\langle 2\rangle)(3-\langle 3\rangle)\rangle$ *extends to higher order truncated correlations?*

(ii) Express $\langle \hat{\rho}(1)|\hat{\rho}(2)|\hat{\rho}(3)\rangle$ *in terms of the truncated correlations by extracting the coincident point contributions :*

$$\langle \hat{\rho}(1)|\hat{\rho}(2)|\hat{\rho}(3)\rangle = \rho(1|2|3) + \delta_{12}\rho(1|3) + \delta_{13}\rho(1|2) + \delta_{23}\rho(1|2) + \delta_{12}\delta_{23}\rho(1) \tag{81}$$

For (i) and (ii), apply the definitions.

Ex. B.2 *The truncated correlations and truncated density averages verify the charge sum rule and the multipolar sum rules in the same form as the correlation functions (see (2.50) e.g. for* $n = 3$

$$\int d\mathbf{r}_1 \sum_{\gamma_1} e_{\gamma_1}\rho(\gamma_1\mathbf{r}_1|\gamma_2\mathbf{r}_2|\gamma_3\mathbf{r}_3) = -(e_{\gamma_2} + e_{\gamma_3})\rho(\gamma_2\mathbf{r}_2|\gamma_3\mathbf{r}_3) \tag{82}$$

$$\int d\mathbf{r}_1 \sum_{\gamma_1} e_{\gamma_1}\langle \hat{\rho}(\gamma_1\mathbf{r}_1)|\hat{\rho}(\gamma_2\mathbf{r}_2)|\hat{\rho}(\gamma_3\mathbf{r}_3)\rangle = 0 \tag{83}$$

The relation (82) follows from (B.9),

$$\rho(1|2|3) = [\rho(1,2,3) - \rho(1)\rho(2,3)] - [\rho(1,2) - \rho(1)\rho(2)]\,\rho(3) - [\rho(1,3) - \rho(1)\rho(3)]\,\rho(2),$$

and from the standard charge sum rule (2.50), where as (83) follows directly from (81) and (82).

Exercices of Appendix C

Ex. C.1 *Deduce the moment formula (C.9) (Wick theorem) of a Gaussian distribution centered at the origin from the general law of formation of cumulants (B.9).*

If odd moments vanish, we have from the rule (B.13) $\langle x_1x_2x_3x_4\rangle = \langle x_1|x_2|x_3|x_4\rangle + \langle x_1|x_2\rangle\langle x_3|x_4\rangle + \langle x_1|x_3\rangle\langle x_2|x_4\rangle + \langle x_2|x_3\rangle\langle x_1|x_4\rangle$, and $\langle x_i|x_j\rangle = \langle x_ix_j\rangle(\langle x_i\rangle = 0)$. Thus the vanishing of the fourth order cumulant $\langle x_1|x_2|x_3|x_4\rangle$ is equivalent with the moment formula (C.9), and so forth for higher order moments.

Exercices of Appendix G

Ex. G.1 *Calculate the kernel* $\langle x_1|\mathrm{e}^{-\beta H_0}|x_2\rangle$.

Introduce the Fourier representation of the configurational basis

$$|\mathbf{x}\rangle = \left(\frac{1}{2\pi}\right)^{3/2} \int \mathrm{d}\mathbf{k}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}\rangle$$

where $|\mathbf{k}\rangle$ are the wave number eigenvectors $|\mathbf{k}\rangle$ of H_0, $H_0|\mathbf{k}\rangle = \frac{\hbar|\mathbf{k}|^2}{2m}|\mathbf{k}\rangle$:

$$\left\langle \mathbf{x}_1|\mathrm{e}^{-\beta H_0}|\mathbf{x}_2\right\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{k}\mathrm{e}^{-\beta\frac{\hbar^2|\mathbf{k}|^2}{2m}}\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}$$

and use the Fourier transform of a Gaussian $\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}\mathbf{k}\mathrm{e}^{-a\frac{|\mathbf{k}|^2}{2}}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{a^{3/2}}\mathrm{e}^{-\frac{|\mathbf{x}|^2}{2a}}$ (see (C.1) and (C.2)).

Ex. G.2 *Show that the distribution (G.6) is normalized to 1.*

Note first the semi-group law

$$\int \mathrm{d}\boldsymbol{\xi}_2 p(\boldsymbol{\xi}_1, s_1|\boldsymbol{\xi}_2, s_2)p(\boldsymbol{\xi}_2, s_2|\boldsymbol{\xi}_3, s_3) = p(\boldsymbol{\xi}_1, s_1|\boldsymbol{\xi}_3, s_3) \tag{84}$$

Indeed, according to (G.2) and (G.5) one can write

$$p(\boldsymbol{\xi}_1, s_1|\boldsymbol{\xi}_2, s_2) = \left\langle \boldsymbol{\xi}_1|\mathrm{e}^{-(s_2-s_1)H_0}|\boldsymbol{\xi}_2\right\rangle$$

(setting $\frac{\hbar^2}{m} = 1$). Using the completeness relation $\int \mathrm{d}\boldsymbol{\xi}|\boldsymbol{\xi} >< \boldsymbol{\xi}| = I$, the r.h.s of (84) equals

$$\int \mathrm{d}\boldsymbol{\xi}_2 \left\langle \boldsymbol{\xi}_1|\mathrm{e}^{-(s_2-s_1)H_0}|\boldsymbol{\xi}_2\right\rangle \left\langle \boldsymbol{\xi}_2|\mathrm{e}^{-(s_3-s_2)H_0}|\boldsymbol{\xi}_3\right\rangle$$

$$= \left\langle \boldsymbol{\xi}_1|\mathrm{e}^{-(s_2-s_1)H_0}\mathrm{e}^{-(s_3-s_2)H_0}|\boldsymbol{\xi}_3\right\rangle = \left\langle \boldsymbol{\xi}_1|\mathrm{e}^{-(s_3-s_1)H_0}|\boldsymbol{\xi}_3\right\rangle = p(\boldsymbol{\xi}_1, s_1|\boldsymbol{\xi}_3, s_3)$$

This implies that the integral of (G.6) on $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n$ equals $(2\pi)^{3/2}p(0,0|0,1) = 1$.

Ex. G.3 *Find the covariance (G.7) of the Brownian bridge process.*

The covariance vanishes if $\mu \neq \nu$ because of rotation invariance. Then it suffices to do the calculation for one component of ξ of $\boldsymbol{\xi}$, using the one dimensional version of (G.5). Write the joint distribution of $\xi(s_1)$ and $\xi(s_2)$ in the standard form (C.3) of a Gaussian distribution in the two dimensional space $\xi = (\xi_1, \xi_2)$, i.e

$$p(0,0|\xi_1, s_1)p(\xi_1, s_1|\xi_2, s_2)p(\xi_2, s_2|0,1) = \frac{(\det A)^{1/2}}{2\pi}\exp\left[-\frac{1}{2}(\xi, A\xi)\right] \tag{85}$$

$$A = \begin{pmatrix} \frac{s_2}{s_1(s_2-s_1)} & -\frac{1}{s_2-s_1} \\ -\frac{1}{s_2-s_1} & \frac{1-s_1}{(1-s_2)(s_2-s_1)} \end{pmatrix} \qquad \det A = \frac{1}{s_1(s_2-s_1)(1-s_2)}$$

$$\text{with inverse } \mathrm{A}^{-1} = \begin{pmatrix} s_1(1-s_1) & s_1(1-s_2) \\ s_1(1-s_2) & s_2(1-s_2) \end{pmatrix} \tag{86}$$

According to (C.7) the average of $\xi_1\xi_2$ with the distribution (85) equals $A^{-1}_{12} = s_1(1-s_2)$ if $s_1 < s_2$, hence the result (G.7), symmetrical in s_1, s_2.

Ex. G.4 *Verify the relation (G.9).*

Write $\min(s_1, s_2) = \theta(s_2 - s_1)s_1 + \theta(s_1 - s_2)s_2$, $\theta(s) = 1, s > 0, \theta(s) = 0, s < 0$ and $\frac{\mathrm{d}}{\mathrm{d}s}\theta(s) = \delta(s)$.

Exercices of Appendix H

Ex. H.1 *Show the invariance property (H.9) of the measure $D_q(\boldsymbol{\mathcal{X}})$.*

Since both processes $\boldsymbol{\mathcal{X}}(s)$ and $\boldsymbol{\mathcal{X}}^{[u]}(s) = \boldsymbol{\mathcal{X}}(s+u) - \boldsymbol{\mathcal{X}}(u)$, u fixed, are Gaussian it suffices to verify that their covariances (H.6) are identical.

Ex. H.2 *Show that the energy does not depend of the choice of the loop parametrization (H.8).*

Note that the Dirac comb is a q-periodic function of its arguments: $\tilde{\delta}(s - s') = \tilde{\delta}(s[\mathrm{mod}]q - s'[\mathrm{mod}]q')$. Since both paths $\boldsymbol{\mathcal{X}}(s)$ and $\boldsymbol{\mathcal{X}}'(s')$ are also q and q'-periodic, the integrands of (H.11) and (H.13) have the same property. Hence the values of the s and s' integrals are the same when s and s' are shifted to $s + u$ and $s' + u$.

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